System Identification

Lecture 8: Time- & Frequency-domain methods

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Pulse response estimation

Periodic signals

$$Z_N \;=\; \{\,(y(k),u(k)),\;\; k=0,\ldots,N-1\,\}. \qquad u(k) \text{ periodic, } u(k)=u(k+N).$$

Assumption: $g(\tau)=0$ for $\tau > \tau_{\text{max}} \leqslant N$.

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix} = \begin{bmatrix} u(0) & u(N-1) & \cdots & u(1) \\ u(1) & u(0) & \cdots & u(2) \\ \vdots & & \ddots & \vdots \\ u(N-1) & u(N-2) & \cdots & u(0) \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(N-1) \end{bmatrix} + \begin{bmatrix} v(0) \\ v(1) \\ \vdots \\ v(N-1) \end{bmatrix}$$

Discrete Fourier Transform

DFT: matrix representation

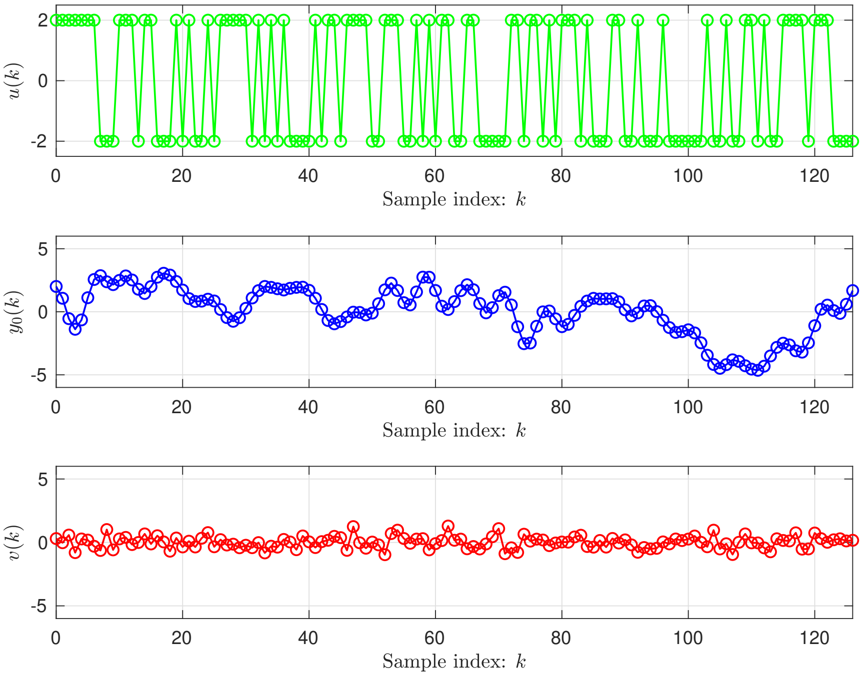
$$Y(\mathrm{e}^{j\omega_n}) = \sum_{k=0}^{N-1} y(k)\mathrm{e}^{-j\omega_n k}, \qquad \omega_n = \frac{2\pi n}{N}, \quad n = 0, \ldots, N-1.$$

$$\begin{bmatrix} Y(\mathrm{e}^{j\omega_0}) \\ Y(\mathrm{e}^{j\omega_1}) \\ \vdots \\ Y(\mathrm{e}^{j\omega_{N-1}}) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \mathrm{e}^{-j\omega_1} & \cdots & \mathrm{e}^{-j\omega_1(N-1)} \\ \vdots & & & \vdots \\ 1 & \mathrm{e}^{-j\omega_{N-1}} & \cdots & \mathrm{e}^{-j\omega_{N-1}(N-1)} \end{bmatrix} \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix}$$

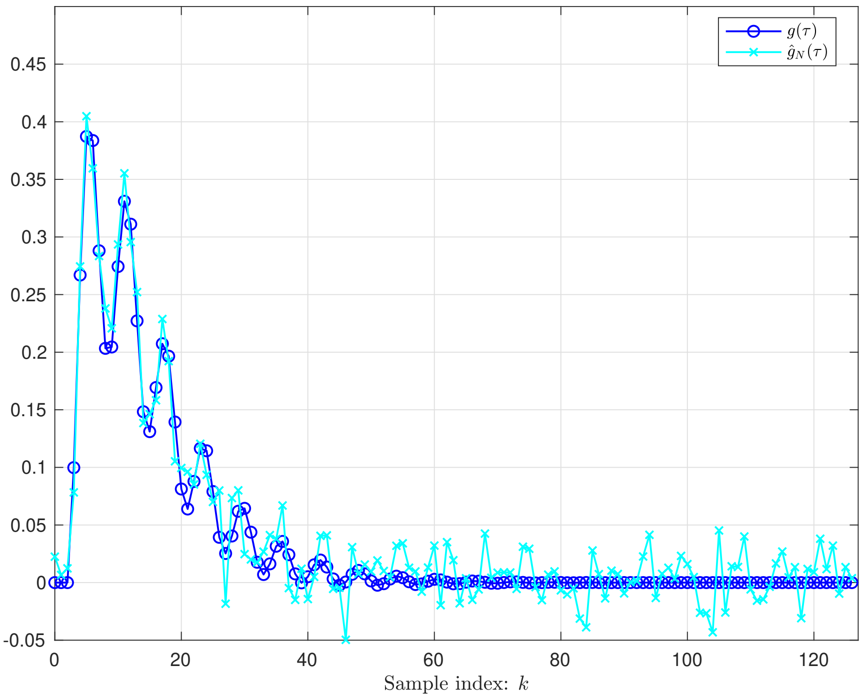
Frequency domain transformation

Transform via matrix representation

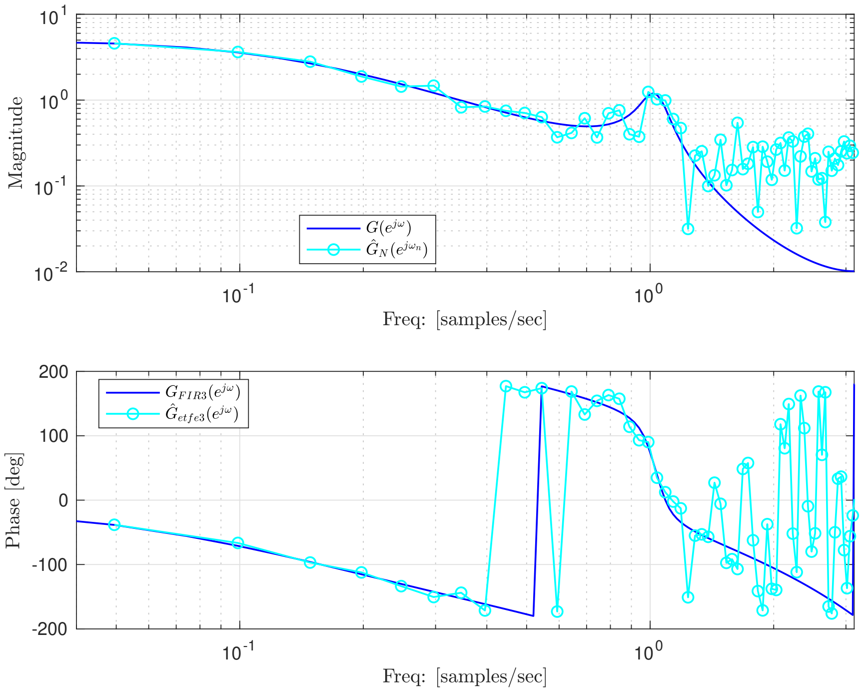
Example



Example



Example

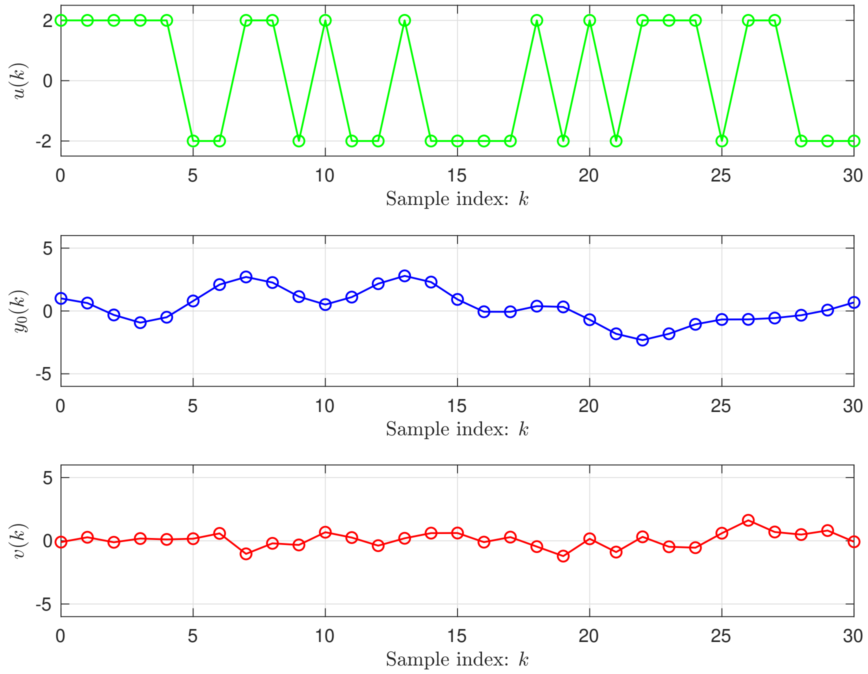


Slowly decaying systems

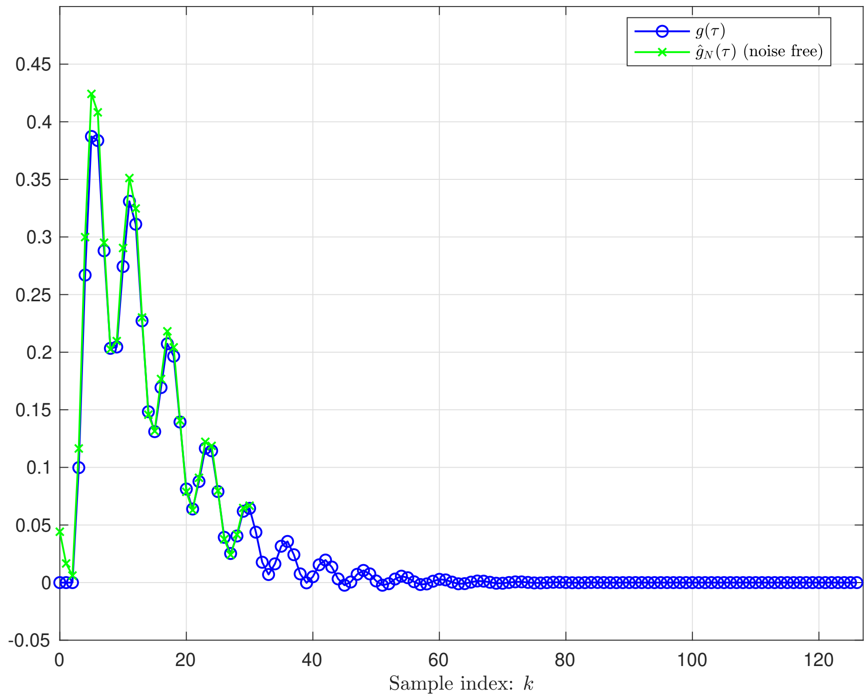
$N < \tau_{\text{max}}$ case

Assume periodic input $u(k) = u(k+N)$.

Example



Example



Slowly decaying systems

$N < \tau_{\mathsf{max}}$ case

$$
\begin{aligned}
y(k) &= \sum_{r=0}^{\infty} \sum_{i=0}^{N-1} T_{u_p}(k,i) g(i+rN) + v(k), \quad k = 0,\ldots,N-1, \\
&= \sum_{i=0}^{N-1} T_{u_p}(k,i) \sum_{r=0}^{\infty} g(i+rN) + v(k), \quad k = 0,\ldots,N-1,
\end{aligned}
$$

Define:

$$
\begin{bmatrix} g_a(0) \\ g_a(1) \\ \vdots \\ g_a(N-1) \end{bmatrix} = \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(N-1) \end{bmatrix} + \begin{bmatrix} g(N) \\ g(N+1) \\ \vdots \\ g(2N-1) \end{bmatrix} + \begin{bmatrix} g(2N) \\ g(2N+1) \\ \vdots \\ g(3N-1) \end{bmatrix} + \cdots
$$

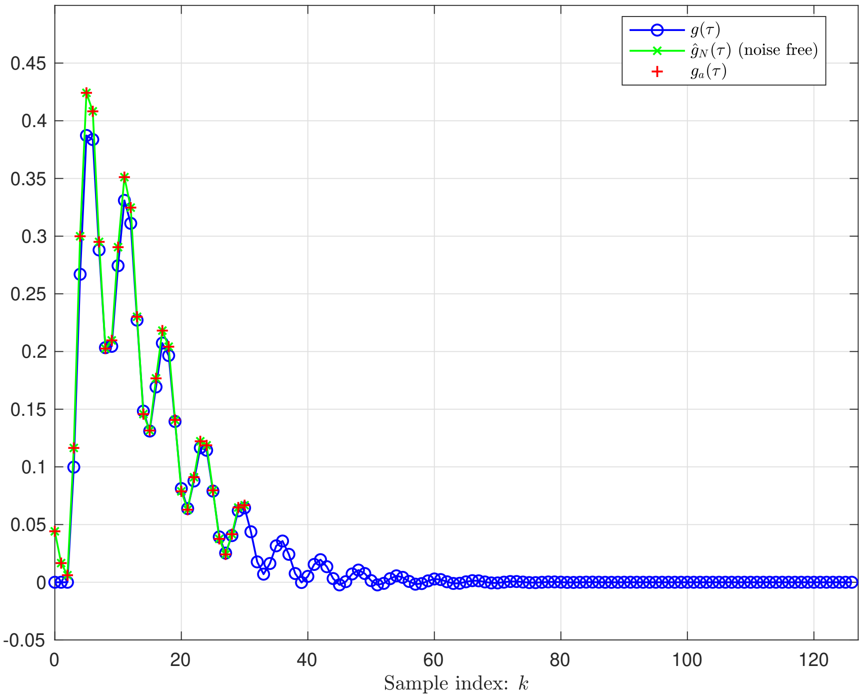
Slowly decaying systems

$N < \tau_{\mathsf{max}}$ case

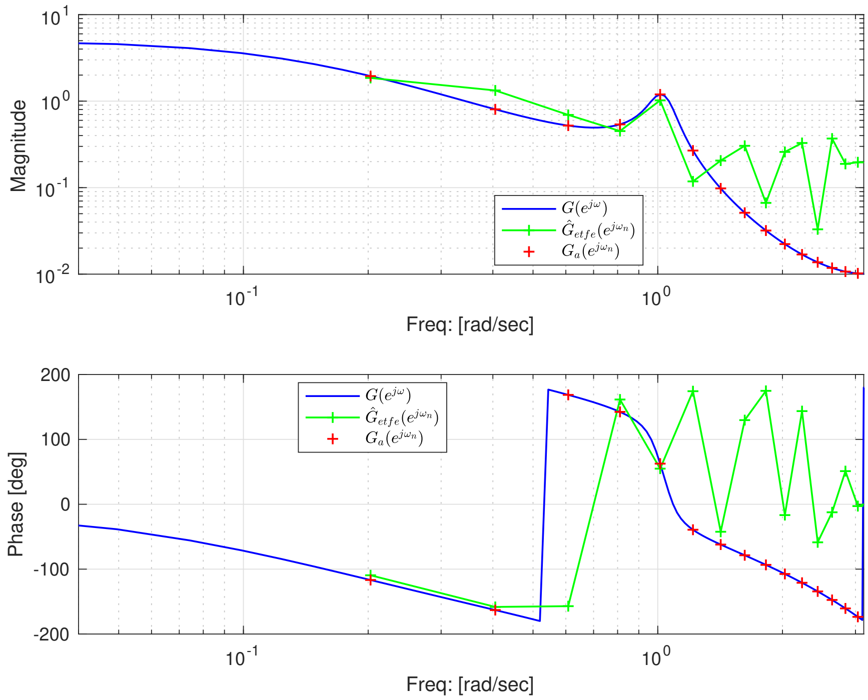
$$
y(k) = \sum_{i=0}^{N-1} T_{u_p}(k,i) g_a(i) + v(k), \qquad k = 0,\ldots,N-1.
$$

$$
y = T_{u_p} g_a + v.
$$

Example



Example



Non-periodic frequency domain estimation

$K \gg N \geqslant \tau_{\text{max}}$ case

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix} = \begin{bmatrix} u(0) & u(-1) & \cdots & u(-N+1) \\ u(1) & u(0) & \cdots & u(-N+2) \\ \vdots & & \ddots & \vdots \\ u(N-1) & u(N-2) & \cdots & u(0) \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(N-1) \end{bmatrix} + \begin{bmatrix} v(0) \\ v(1) \\ \vdots \\ v(N-1) \end{bmatrix}$$

Non-periodic frequency domain estimation

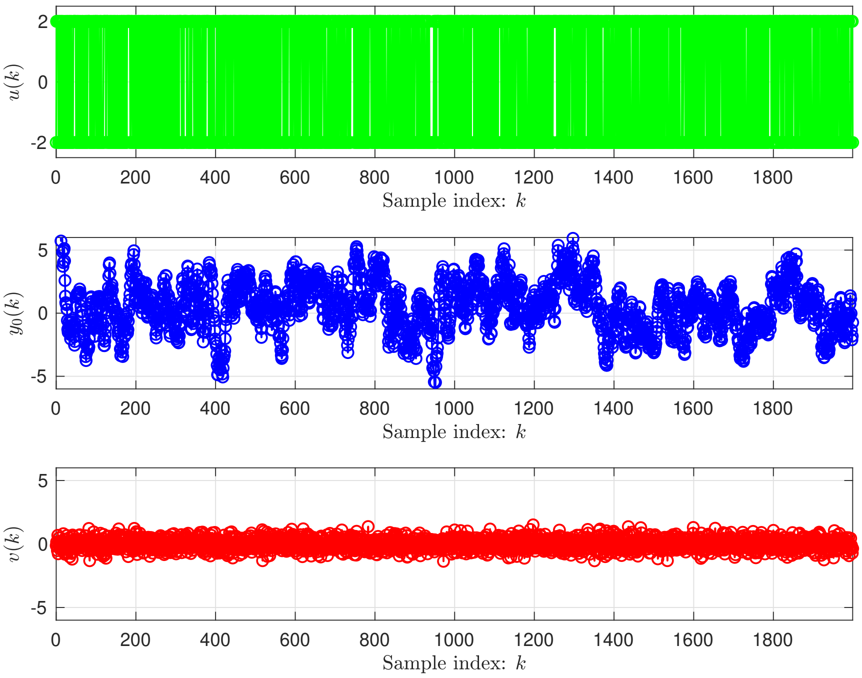
Bias and variance

$$\hat{G}(\mathrm{e}^{j\omega_n}) \;=\; G_0(\mathrm{e}^{j\omega_n}) \;+\; X_u^{-1} V(\mathrm{e}^{j\omega_n})$$

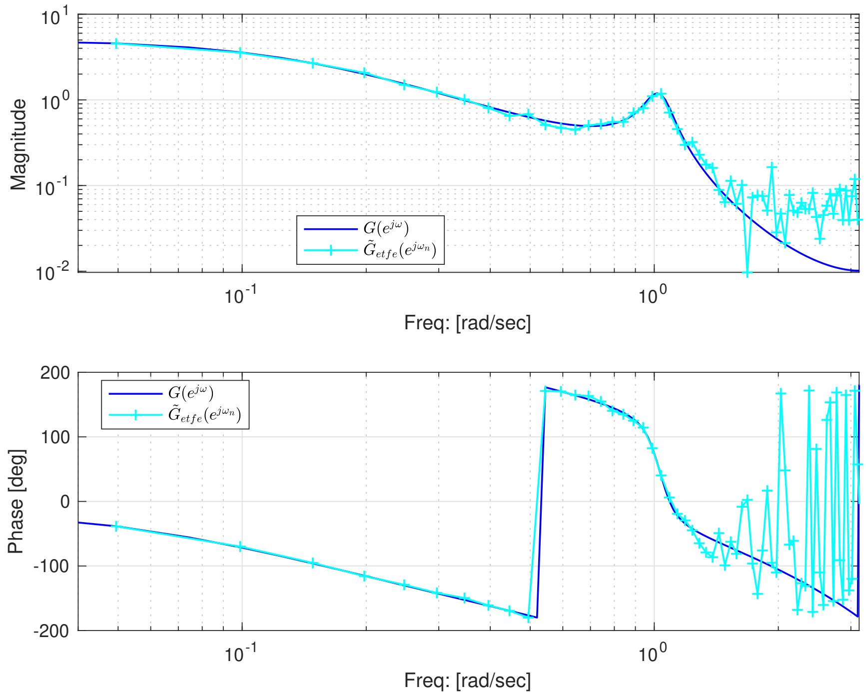
Very long data records

Averaging application

Example



Example



Time- and frequency-domain methods