System Identification

Supplementary notes: lecture 6

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6 Spectral estimation, smoothing, & input signals

6.1 Spectral estimation: auto-spectra

The input-output relationship between the auto-spectra,

$$\begin{aligned}\phi_y(\mathrm{e}^{j\omega}) \;=\; & G(\mathrm{e}^{j\omega})\phi_u(\mathrm{e}^{j\omega})G^*(\mathrm{e}^{j\omega}) + \phi_v \\ & + G(\mathrm{e}^{j\omega})\phi_{uv}(\mathrm{e}^{j\omega}) + \phi_{vu}(\mathrm{e}^{j\omega})G^*(\mathrm{e}^{j\omega}),\end{aligned}$$

is relatively easy to show.

Begin with the noise-free case. We do this for the multivariable case as it makes generalising to the noisy case easier. To avoid writing out the indices for the input-output components we must interpret $g(m)$, etc. as a matrix and $u(k)$, etc., as vectors.

$$\begin{aligned}\phi_y(\mathrm{e}^{j\omega}) &= \sum_{\tau=-\infty}^{\infty} E\{y(k)y^T(k-\tau)\}\mathrm{e}^{-j\omega\tau} \\ &= \sum_{\tau=-\infty}^{\infty} E\left\{\left(G(\mathrm{e}^{j\omega})u(k)\right)\left(G(\mathrm{e}^{j\omega})u(k-\tau)\right)^T\right\}\mathrm{e}^{-j\omega\tau} \\ &= \sum_{\tau=-\infty}^{\infty} E\left\{\left(\sum_{l=-\infty}^{\infty} g(l)u(k-l)\right)\left(\sum_{m=-\infty}^{\infty} u^T(k-\tau-m)g^T(m)\right)\right\}\mathrm{e}^{-j\omega\tau} \\ &= \sum_{l=-\infty}^{\infty} g(l) \sum_{m=-\infty}^{\infty}\sum_{\tau=-\infty}^{\infty} E\{u(k-l)u^T(k-\tau-m)\}g(m)\mathrm{e}^{-j\omega\tau} \\ &= \sum_{l=-\infty}^{\infty} g(l)\mathrm{e}^{-j\omega l} \sum_{m=-\infty}^{\infty}\sum_{\tau=-\infty}^{\infty} E\{u(k-l)u^T(k-\tau-m)\}\mathrm{e}^{-j\omega(\tau-l+m)}g^T(m)\mathrm{e}^{j\omega m}\end{aligned}$$

and by substituting $s = k - l$,

$$= \sum_{l=-\infty}^{\infty} g(l)\mathrm{e}^{-j\omega l} \sum_{m=-\infty}^{\infty} \sum_{\tau=-\infty}^{\infty} E\{u(s)u^T(s+l-\tau-m)\}\mathrm{e}^{-j\omega(\tau-l+m)} g^T(m)\mathrm{e}^{j\omega m}$$

and by substituting $t = \tau - l + m$,

$$\begin{aligned}
&= \underbrace{\sum_{l=-\infty}^{\infty} g(l)\mathrm{e}^{-j\omega l}}_{G(\mathrm{e}^{j\omega})} \; \underbrace{\sum_{\tau=-\infty}^{\infty} E\{u(s)u^T(s-t)\}\mathrm{e}^{-j\omega t}}_{\phi_u(\mathrm{e}^{j\omega})} \; \underbrace{\sum_{m=-\infty}^{\infty} g^T(m)\mathrm{e}^{j\omega m}}_{G^T(\mathrm{e}^{-j\omega})} \\
&= G(\mathrm{e}^{j\omega})\, \phi_u(\mathrm{e}^{j\omega})\, G^T(\mathrm{e}^{-j\omega}) \\
&= G(\mathrm{e}^{j\omega})\, \phi_u(\mathrm{e}^{j\omega})\, G^*(\mathrm{e}^{j\omega})
\end{aligned}$$

To extend this to the system plus noise use,

$$\begin{aligned}
y(k) &= G(\mathrm{e}^{j\omega})u(k) + v(k) \\
&= \begin{bmatrix} G(\mathrm{e}^{j\omega}) & I \end{bmatrix} \begin{bmatrix} u(k) \\ v(k) \end{bmatrix}.
\end{aligned}$$

Then,

$$\begin{aligned}
\phi_y(\mathrm{e}^{j\omega}) &= \begin{bmatrix} G(\mathrm{e}^{j\omega}) & I \end{bmatrix} \begin{bmatrix} \phi_u(\mathrm{e}^{j\omega}) & \phi_{uv}(\mathrm{e}^{j\omega}) \\ \phi_{vu}(\mathrm{e}^{j\omega}) & \phi_v(\mathrm{e}^{j\omega}) \end{bmatrix} \begin{bmatrix} G^*(\mathrm{e}^{j\omega}) \\ I \end{bmatrix} \\
&= G(\mathrm{e}^{j\omega})\phi_u(\mathrm{e}^{j\omega})G^*(\mathrm{e}^{j\omega}) + G(\mathrm{e}^{j\omega})\phi_{uv}(\mathrm{e}^{j\omega}) + \phi_{vu}(\mathrm{e}^{j\omega})G^*(\mathrm{e}^{j\omega}) + \phi_v.
\end{aligned}$$

When $u(k)$ and $v(k)$ are uncorrelated we have $\phi_{uv}(\mathrm{e}^{j\omega}) = 0$ and the simplifications given in the slide immediately follow.

Note that only in the SISO case do we have,

$$G(\mathrm{e}^{j\omega})\phi_{uv}(\mathrm{e}^{j\omega}) + \phi_{vu}(\mathrm{e}^{j\omega})G^*(\mathrm{e}^{j\omega}) = 2\,\mathrm{real}\left\{G(\mathrm{e}^{j\omega})\phi_{uv}(\mathrm{e}^{j\omega})\right\}.$$

See also [1, p. 45].

6.2 Spectral estimation: cross-spectra

The cross-spectral input-output relationship for our system is,

$$\phi_{yu}(\mathrm{e}^{j\omega}) \;=\; G(\mathrm{e}^{j\omega})\, \phi_u(\mathrm{e}^{j\omega}) + \phi_{uv}(\mathrm{e}^{j\omega}).$$

To show this we first consider the $v(k) = 0$ (noise-free) result,

$$\begin{aligned}
\phi_{yu}(\mathrm{e}^{j\omega}) &= \sum_{\tau=-\infty}^{\infty} E\{y(k)u^T(k-\tau)\}\mathrm{e}^{-j\omega\tau} \\
&= \sum_{\tau=-\infty}^{\infty} E\left\{\left(\sum_{l=-\infty}^{\infty} g(l)u(k-l)\right) u^T(k-\tau)\right\} \mathrm{e}^{-j\omega\tau} \\
&= \sum_{l=-\infty}^{\infty} g(l)\mathrm{e}^{-j\omega l} \sum_{\tau=-\infty}^{\infty} E\{u(k-l)u^T(k-\tau)\}\mathrm{e}^{-j\omega(\tau-l)}
\end{aligned}$$

and by substituting $s = k - l$,

$$= G(\mathrm{e}^{j\omega}) \sum_{\tau=-\infty}^{\infty} E\{u(s)u^T(s+l-\tau)\}\mathrm{e}^{-j\omega(\tau-l)}$$

and by substituting $t = \tau - l$,

$$\begin{aligned}
&= G(\mathrm{e}^{j\omega}) \underbrace{\sum_{\tau=-\infty}^{\infty} E\{u(s)u^T(s-t)\}\mathrm{e}^{-j\omega t}}_{\phi_u(\mathrm{e}^{j\omega})} \\
&= G(\mathrm{e}^{j\omega})\,\phi_u(\mathrm{e}^{j\omega})
\end{aligned}$$

The general ($v(k) \neq 0$) result is the MIMO application of the above to,

$$y(k) = \begin{bmatrix} G(\mathrm{e}^{j\omega}) & I \end{bmatrix} \begin{bmatrix} u(k) \\ v(k) \end{bmatrix}.$$

6.3 Time-domain window functions

There is a subtlety in the construction of the time-domain window, $w_\gamma(\tau)$. Consider a Bartlett window defined by the width parameter γ,

$$w_\gamma(\tau) = \begin{cases} 0 & \tau < -\gamma \\ 1 - \frac{|\tau|}{\gamma} & -\gamma \leqslant \tau \leqslant \gamma \\ 0 & \tau > \gamma. \end{cases}$$

The window has a peak value of 1 at $\tau = 0$. Examine what happens when we choose $\gamma = N/2$. In this case

$$w_{\gamma=N/2}(-N/2) = 0 \quad \text{and} \quad w_{\gamma=N/2}(N/2) = 0.$$

Careful counting reveals that for N even the window has $N/2 - 1$ non-zero values[1].

[1]For some windows, such as a Hamming window, the edge of the window is defined with a non-zero value. In this case a $\gamma = N/2$ width window will have $N/2 + 1$ non-zero values.

The correct interpretation is that the window is of length N and includes (at either the beginning or the end) a single zero value. This definition makes the Bartlett window a triangular waveform of periodicity N. And from the Fourier series of a triangular waveform we know that it has spectral components,

$$\omega = \frac{2\pi}{N}, 0, \frac{3 \times 2\pi}{N}, 0, \frac{5 \times 2\pi}{N}, 0, \ldots$$

If we had defined our Bartlett window to have two zeros (one at the start and one at the end) within the length N window then it would not exactly correspond to a triangular waveform and would have a slightly "messier" spectrum.

The reason that this will make a difference is that we will see that the spectrum that we will measure at the N DFT frequencies ($2\pi n/N$, $n = 0, 1, \ldots, N/2$) is the convolution of the frequency responses of the window function and the underlying signal.

References

[1] L. Ljung, *System Identification: Theory for the User*, 2nd ed. Prentice-Hall, 1999.