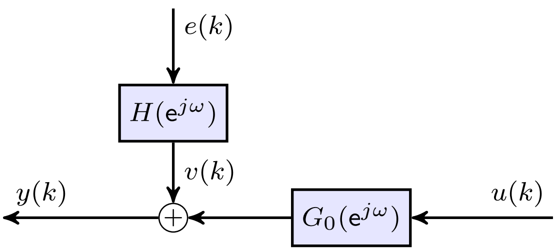
System Identification

Lecture 6: Frequency-domain identification & input signals

Roy Smith

Spectral transformations (random signals)



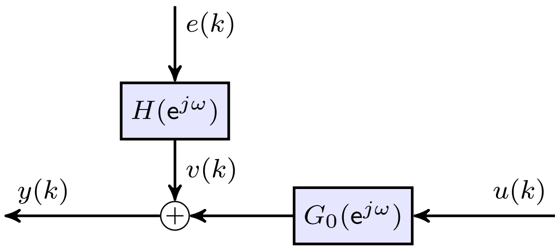
If $v(k) = 0$ then,

$$\phi_y(\mathrm{e}^{j\omega}) = \left|G(\mathrm{e}^{j\omega})\right|^2 \phi_u(\mathrm{e}^{j\omega}).$$

If $v(k) \neq 0$ then,

$$\begin{aligned}\phi_y(\mathrm{e}^{j\omega}) &= |G(\mathrm{e}^{j\omega})|^2\phi_u(\mathrm{e}^{j\omega}) + \phi_v + 2\,\mathrm{real}\, G(\mathrm{e}^{j\omega})\phi_{uv}(\mathrm{e}^{j\omega}) \\ &= |G(\mathrm{e}^{j\omega})|^2\phi_u(\mathrm{e}^{j\omega}) + \phi_v, \quad \text{(if uncorrelated)} \\ &= |G(\mathrm{e}^{j\omega})|^2\phi_u(\mathrm{e}^{j\omega}) + |H(\mathrm{e}^{j\omega})|^2\end{aligned}$$

Spectral transformations (random signals)



Cross-correlation result:

$$\begin{aligned}\phi_{yu}(\mathrm{e}^{j\omega}) &= G(\mathrm{e}^{j\omega})\,\phi_u(\mathrm{e}^{j\omega}) + \phi_{uv}(\mathrm{e}^{j\omega})\\ &= G(\mathrm{e}^{j\omega})\,\phi_u(\mathrm{e}^{j\omega})\end{aligned}$$

(if $u(k)$ and $v(k)$ are uncorrelated)

Spectral estimation methods:

$$\hat{G}(\mathrm{e}^{j\omega}) = \frac{\hat{\phi}_{yu}(\mathrm{e}^{j\omega})}{\hat{\phi}_u(\mathrm{e}^{j\omega})}$$

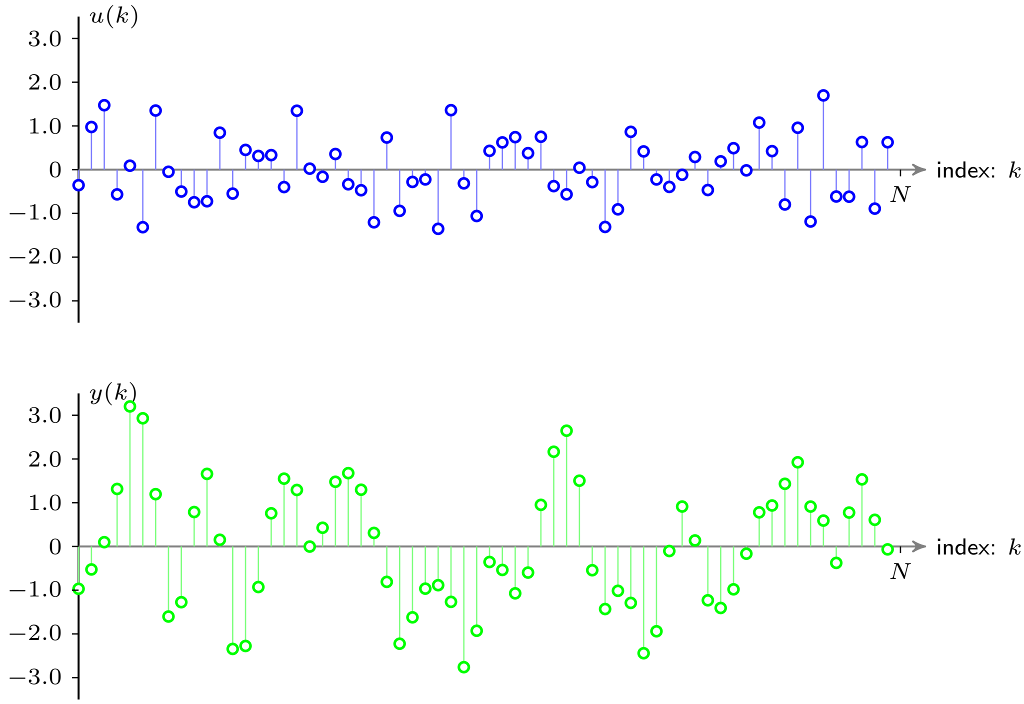
Spectral estimation methods

$$\phi_y(\mathrm{e}^{j\omega}) = |G(\mathrm{e}^{j\omega})|^2\phi_u(\mathrm{e}^{j\omega}) + \phi_v(\mathrm{e}^{j\omega})$$

$$\phi_{yu}(\mathrm{e}^{j\omega}) = G(\mathrm{e}^{j\omega})\,\phi_u(\mathrm{e}^{j\omega})$$

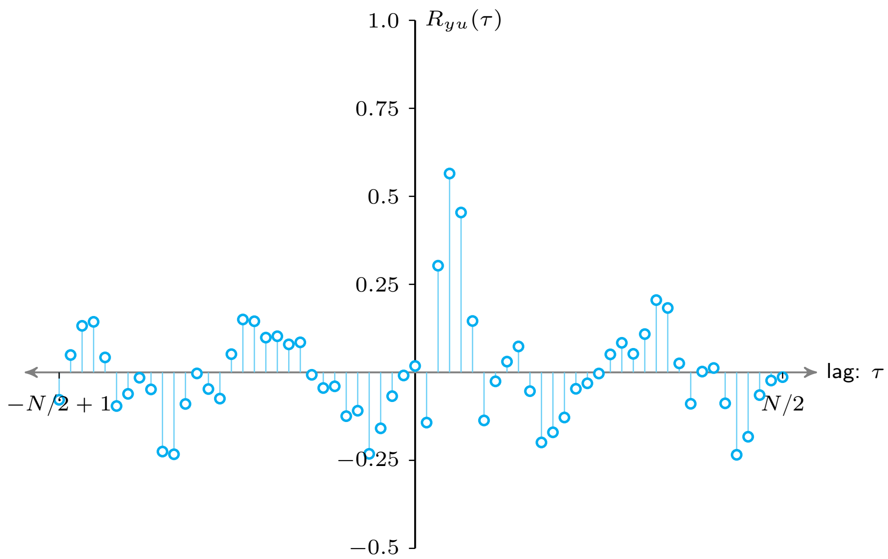
$$\hat{G}(\mathrm{e}^{j\omega_n}) = \frac{\hat{\phi}_{yu}(\mathrm{e}^{j\omega_n})}{\hat{\phi}_u(\mathrm{e}^{j\omega_n})}$$

Spectral analysis example (periodic signal, $N = M$)



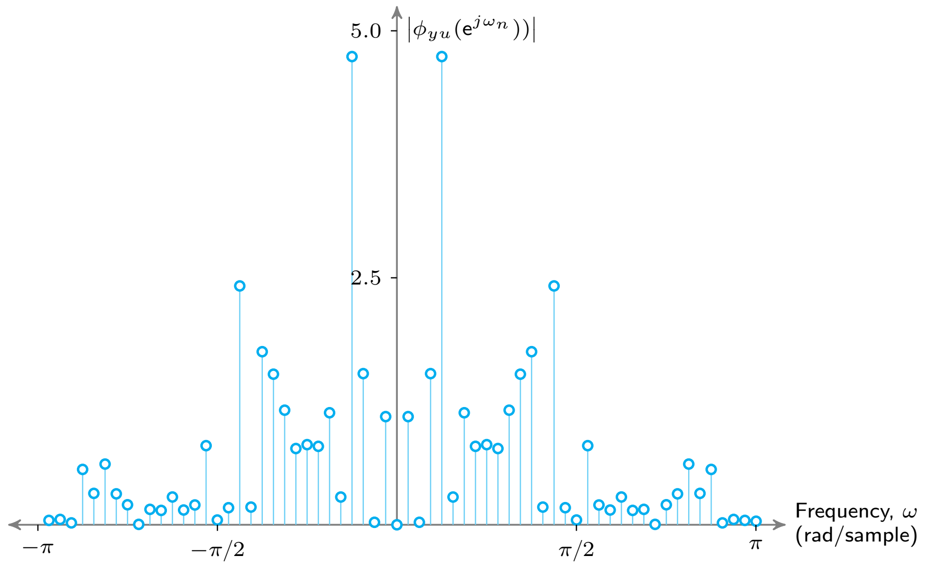
Estimate of the cross-correlation function (periodic signal, $N = M$)

$$R_{yu}(\tau) = \frac{1}{N} \sum_{k=0}^{N-1} y(k)u(k-\tau), \quad \tau = -N/2 + 1, \ldots, N/2.$$



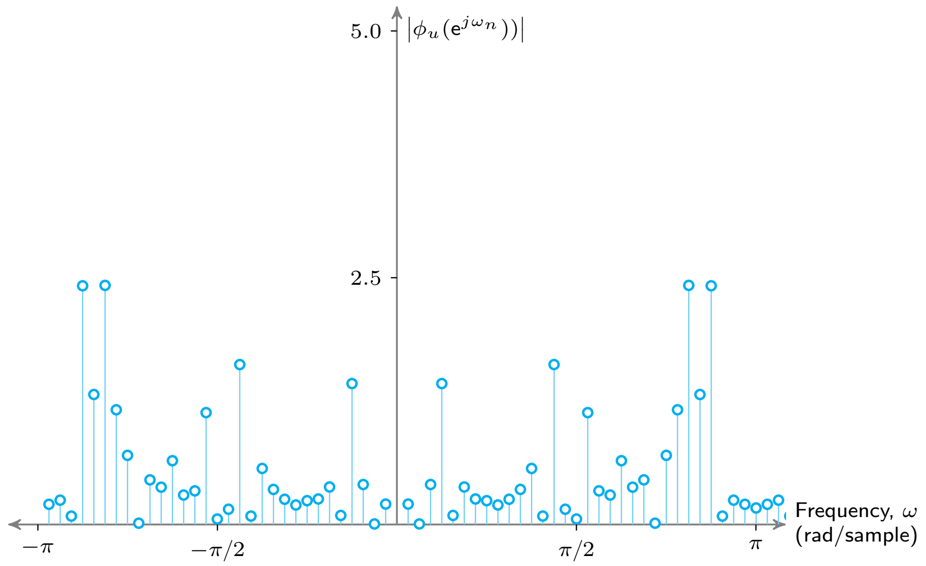
Estimate of the cross power spectral density (periodic signal, $N = M$)

$$\phi_{yu}(\mathrm{e}^{j\omega_n}) = \sum_{\tau=0}^{N-1} R_{yu}(\tau)\mathrm{e}^{-j\omega_n \tau}, \quad \omega_n = \frac{2\pi n}{N}, \quad n = -N/2+1, \ldots, N/2.$$



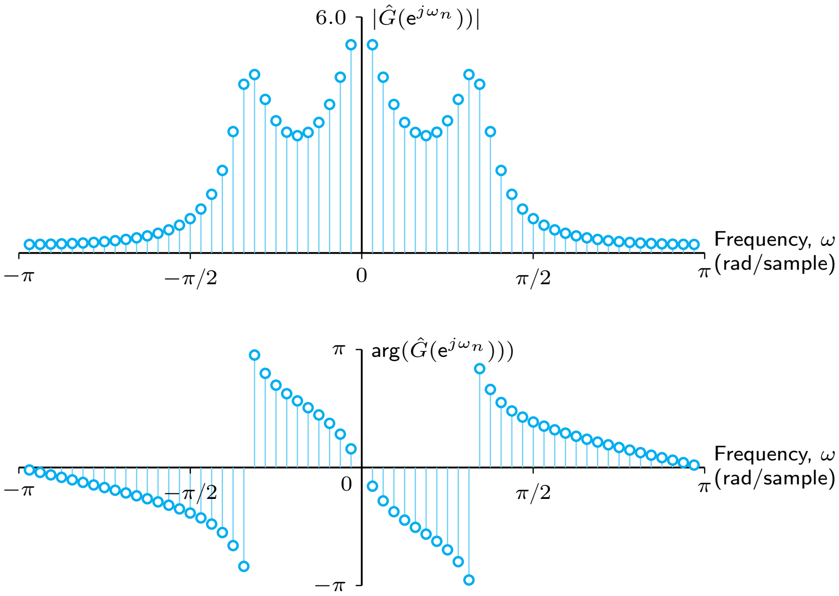
Estimate of the input power spectral density (periodic signal, $N = M$)

$$\phi_{u}(\mathrm{e}^{j\omega_n}) = \sum_{\tau=0}^{N-1} R_{u}(\tau)\mathrm{e}^{-j\omega_n \tau}, \quad \omega_n = \frac{2\pi n}{N}, \quad n = -N/2+1, \ldots, N/2.$$



Estimated transfer function

$$\hat{G}(e^{j\omega_n}) \;=\; \phi_{yu}(e^{j\omega_n}) \,/\, \phi_u(e^{j\omega_n})$$



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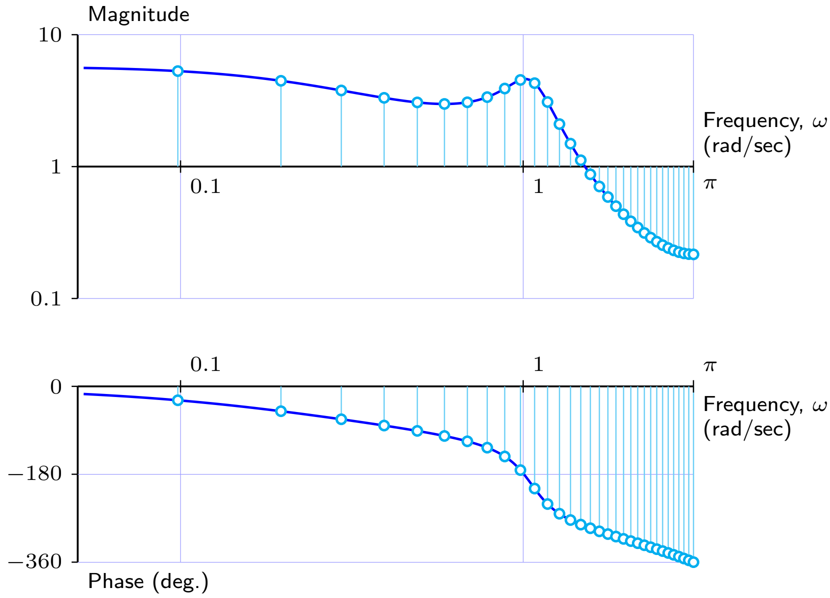


Spectral estimation (periodic signals)

Autocorrelation function

Periodic signal, $x(k)$, with period M
($N = mM$ for some integer m)

$$R_x(\tau) = \frac{1}{M} \sum_{k=0}^{M-1} x(k)x(k-\tau) \qquad \text{(using a periodic calculation)}$$

Power spectral density

The power spectral density can be calculated exactly and is also equal to the periodogram.

$$\phi_x(\mathrm{e}^{j\omega_n}) = \sum_{\tau=0}^{M-1} R_x(\tau)\mathrm{e}^{-j\omega_n\tau} = \frac{1}{M}|X_M(\mathrm{e}^{j\omega_n})|^2$$

Spectral estimation of random signals

Periodogram

The periodogram (for a length N measurement of a random signal $v(k)$) is:

$$\frac{1}{N}\left|V_N(\mathrm{e}^{j\omega})\right|^2$$

Asymptotically unbiased estimator of the spectrum:

$$\lim_{N\longrightarrow\infty} E\left\{\frac{1}{N}|V_N(\mathrm{e}^{j\omega})|^2\right\} = \phi_v(\mathrm{e}^{j\omega})$$

Assumes:

$$\lim_{N\longrightarrow\infty} \frac{1}{N} \sum_{\tau=-N}^{N} |\tau R_v(\tau)| = 0$$

Spectral estimation (via covariances)

Autocovariance function

Autocovariance estimate (stochastic, zero mean, $v(k)$):

$$\hat{R}_v(\tau) = \begin{cases} \dfrac{1}{N-|\tau|} \displaystyle\sum_{k=\tau}^{N-1} v(k)v(k-\tau), & \text{for } \tau \geqslant 0, \\ \dfrac{1}{N-|\tau|} \displaystyle\sum_{k=0}^{N+\tau-1} v(k)v(k-\tau), & \text{for } \tau < 0, \end{cases}$$

Gives estimates for $-N+1 \leqslant \tau \leqslant N-1$.

This is an unbiased estimator of $R_v(\tau)$: $\quad E\{\hat{R}_v(\tau)\} = R_v(\tau)$

Spectral estimate

$$\hat{\phi}_v(e^{j\omega}) = \sum_{\tau=-N+1}^{N-1} \hat{R}_v(\tau)e^{-j\omega\tau}$$

Spectral estimation (more general case)

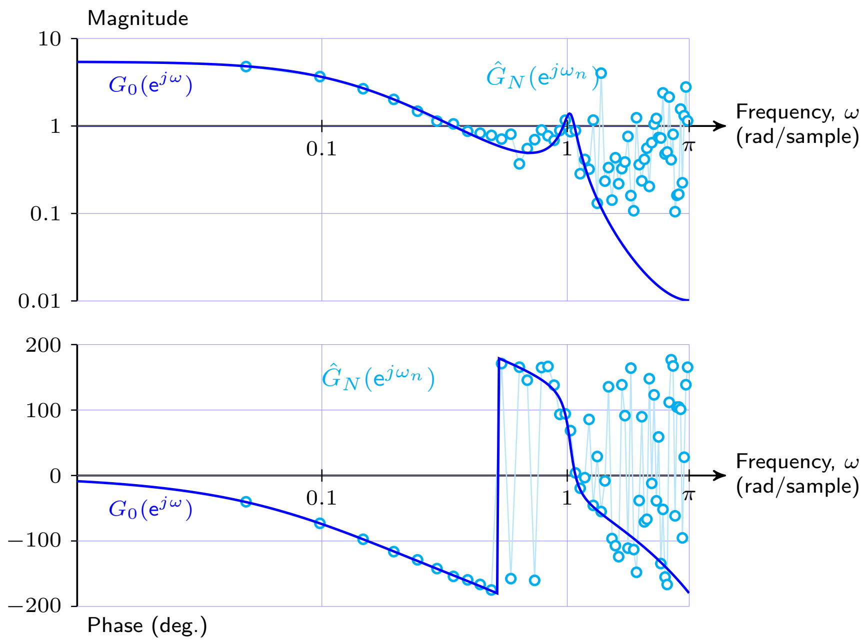
Alternative autocorrelation estimate:

$$\hat{R}_x(\tau) = \begin{cases} \dfrac{1}{N} \displaystyle\sum_{k=\tau}^{N-1} x(k)x(k-\tau), & \text{for } \tau \geqslant 0, \\ \dfrac{1}{N} \displaystyle\sum_{k=0}^{N+\tau-1} x(k)x(k-\tau), & \text{for } \tau < 0, \end{cases}$$

Periodic $x(k)$: unbiased (exact) if $N = mM$ (for integer m)

Random $x(k)$: biased $E\{\hat{R}_x(\tau)\} = \dfrac{N-|\tau|}{N} R_x(\tau)$
asymptotically unbiased
(as $N \longrightarrow \infty$, $\tau/N \longrightarrow 0$)

ETFE smoothing: motivation



ETFE smoothing

Averaging adjacent frequencies

$$E\left\{\left(\hat{G}_N(e^{j\omega_n}) - G_0(e^{j\omega_n})\right)\left(\hat{G}_N(e^{-j\omega_i}) - G_0(e^{-j\omega_i})\right)\right\} = 0$$

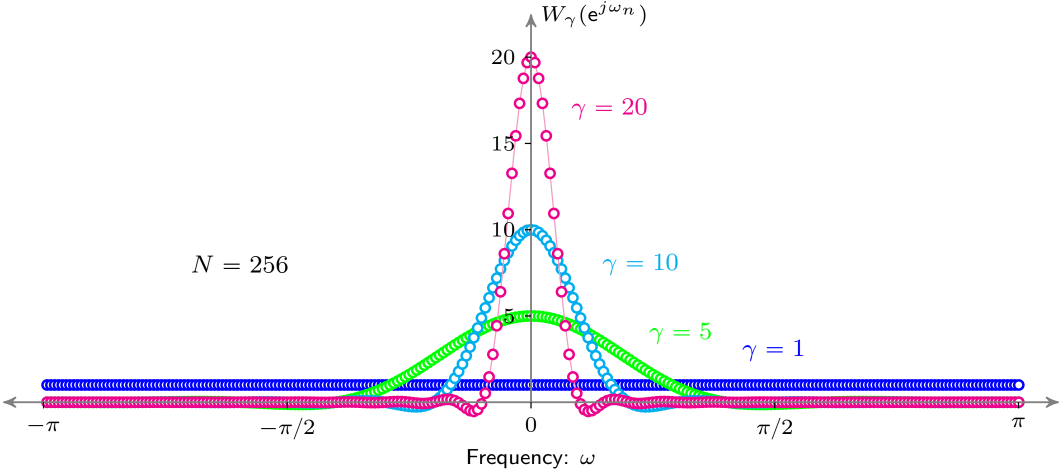
ETFE smoothing

Sliding window

ETFE smoothing

Window function

$W_\gamma(\mathrm{e}^{j\omega_n}), \qquad \gamma$ is a width parameter



This is a Hann window.

Window functions

Frequency smoothing window characteristics: width (specified by γ parameter)

The wider the frequency window (i.e. decreasing γ) ...

- the more adjacent frequencies included in the smoothed estimate,
- the smoother the result,
- the lower the noise induced variance,
- the higher the bias.

Window functions

Window characteristics: shape

Some common choices:

Bartlett: $$W_\gamma(e^{j\omega}) = \frac{1}{\gamma}\left(\frac{\sin \gamma\omega/2}{\sin \omega/2}\right)^2$$

Hann: $$W_\gamma(e^{j\omega}) = \tfrac{1}{2}D_\gamma(\omega) + \tfrac{1}{4}D_\gamma(\omega - \pi/\gamma) + \tfrac{1}{4}D_\gamma(\omega + \pi/\gamma)$$

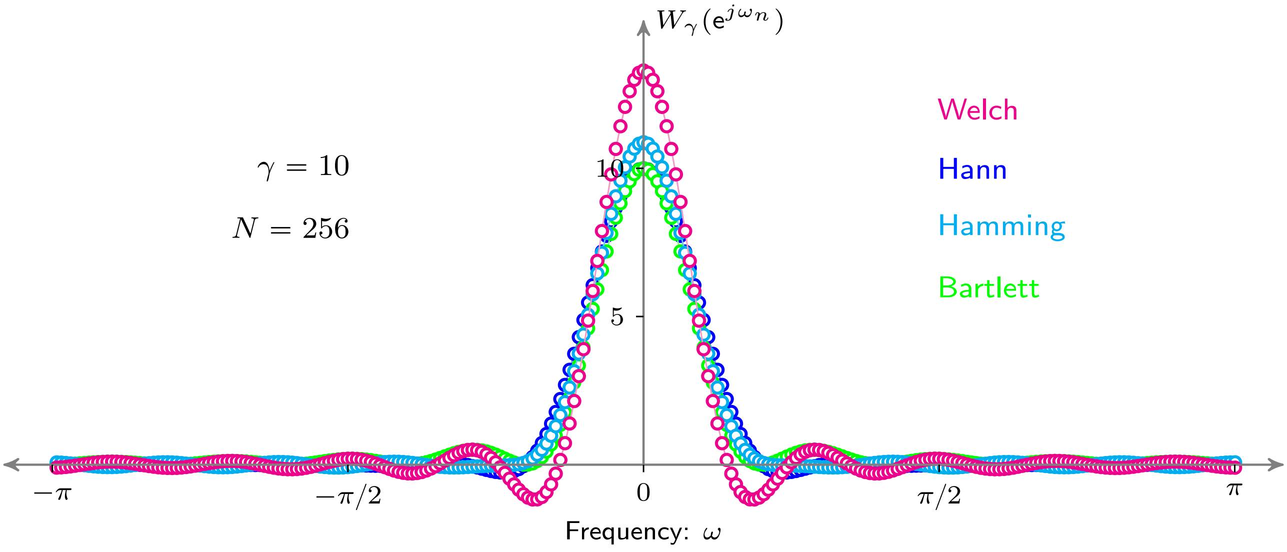
where $$D_\gamma(\omega) = \frac{\sin \omega(\gamma + 0.5)}{\sin \omega/2}$$

Others include: Hamming, Parzen, Kaiser, ...

The differences are mostly in the leakage properties of the energy to adjacent frequencies. And the ability to resolve close frequency peaks.

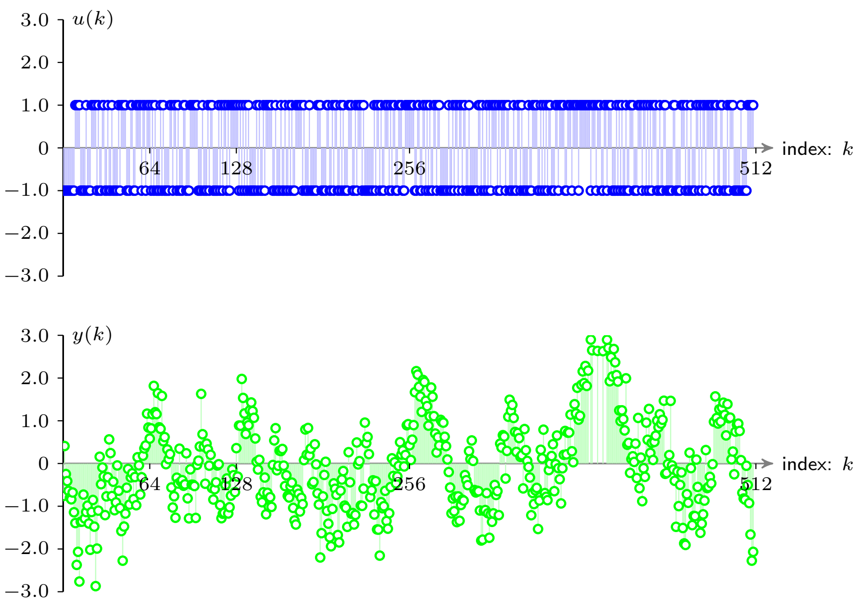
Window functions

Example frequency domain windows.



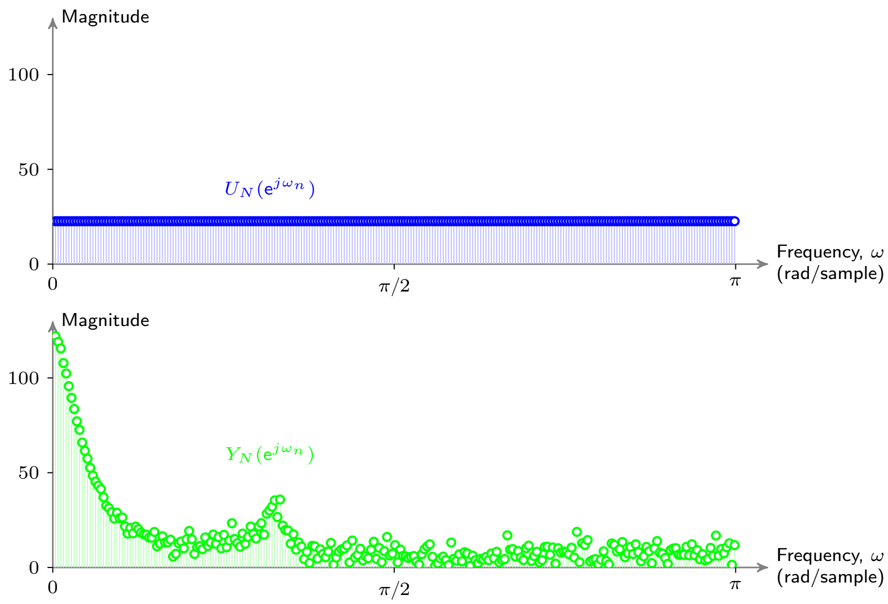
ETFE smoothing example

Experiment: $N = 511$ (periodic experiment)



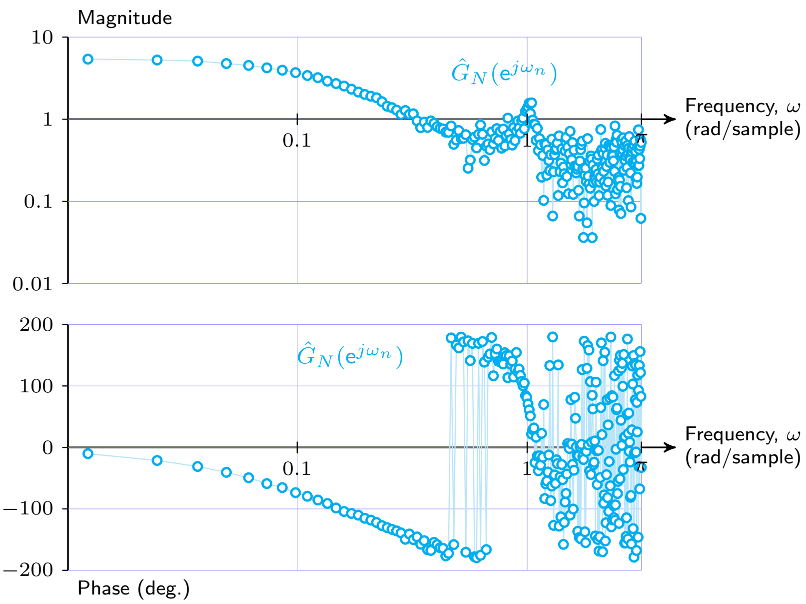
ETFE smoothing example

DFT calculations:



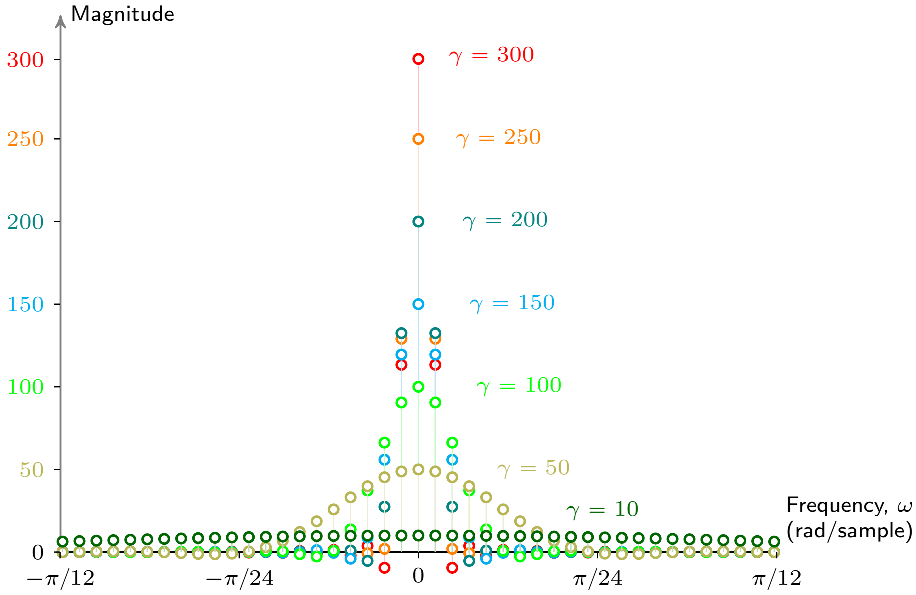
ETFE smoothing example

Raw ETFE: $\hat{G}_N(\mathrm{e}^{j\omega_n}), \quad n = 0, \ldots, N-1, \quad N = 511$



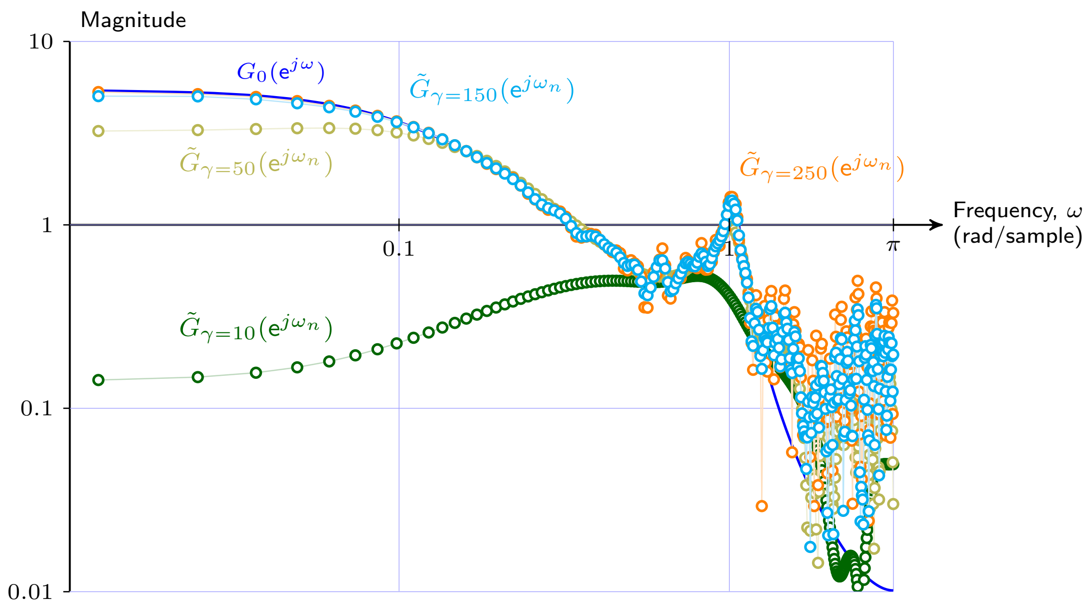
ETFE smoothing example

Frequency domain smoothing windows (Hann):



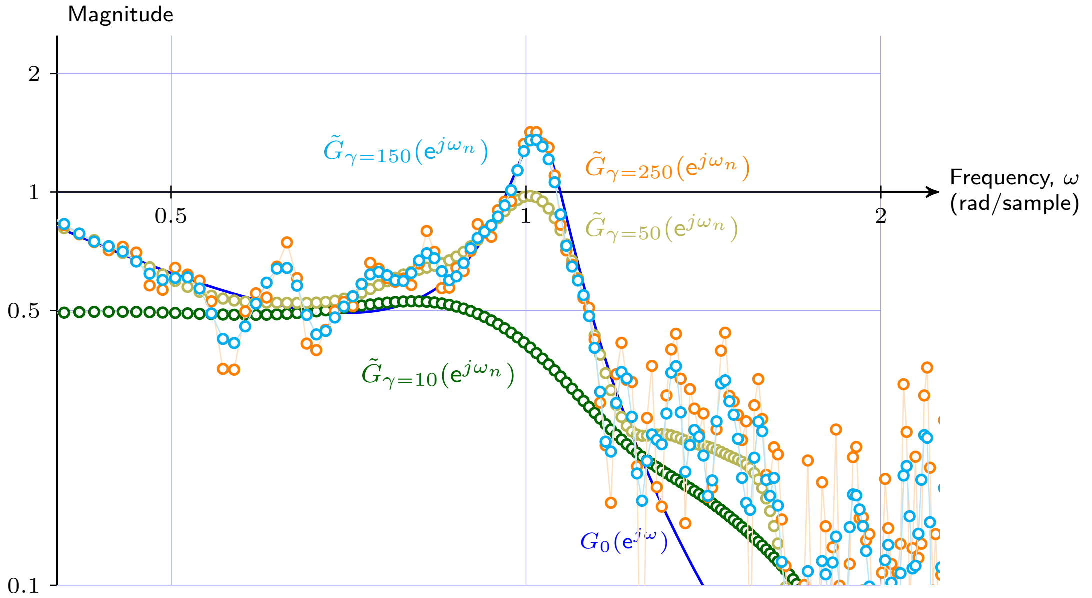
ETFE smoothing example

Smoothed ETFE (for various γ values):



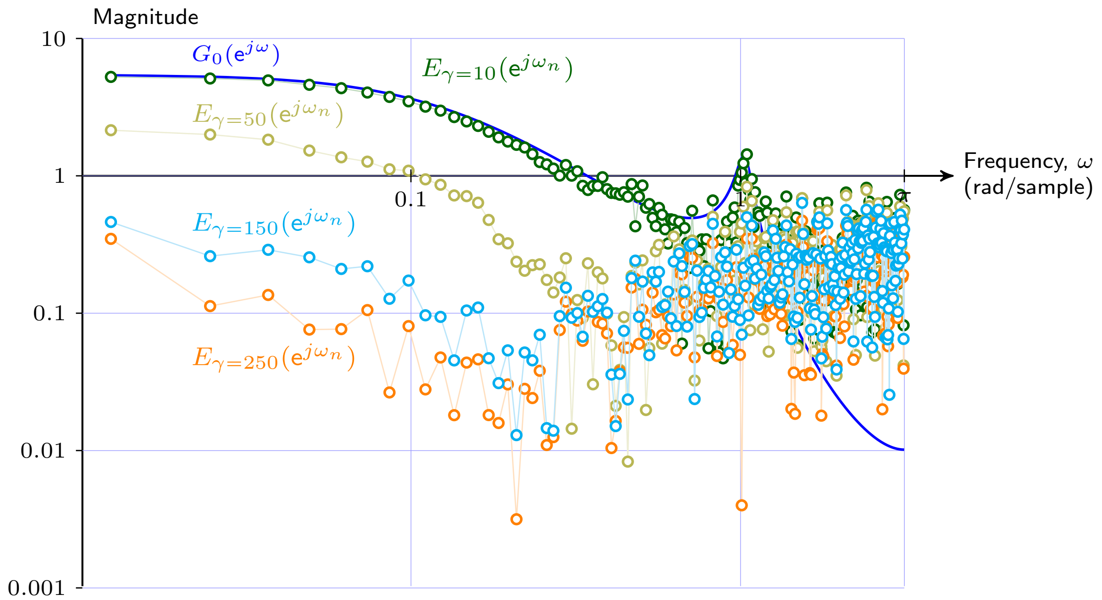
ETFE smoothing example

Smoothed ETFE (for various γ values):



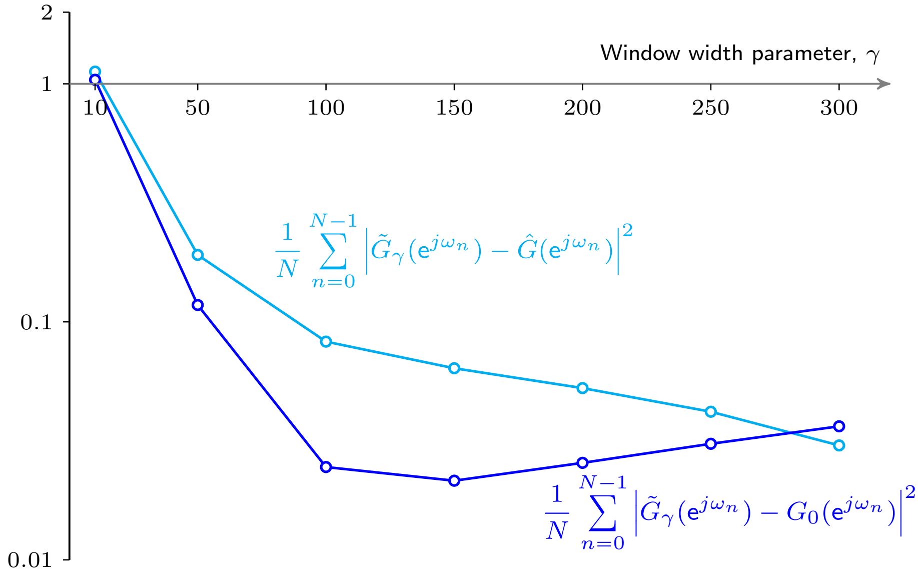
ETFE smoothing example

Absolute fit error: $E_\gamma(e^{j\omega_n}) \; = \; \left|\tilde{G}_\gamma(e^{j\omega_n}) - G_0(e^{j\omega_n})\right|, \quad n = 0, \ldots, N$



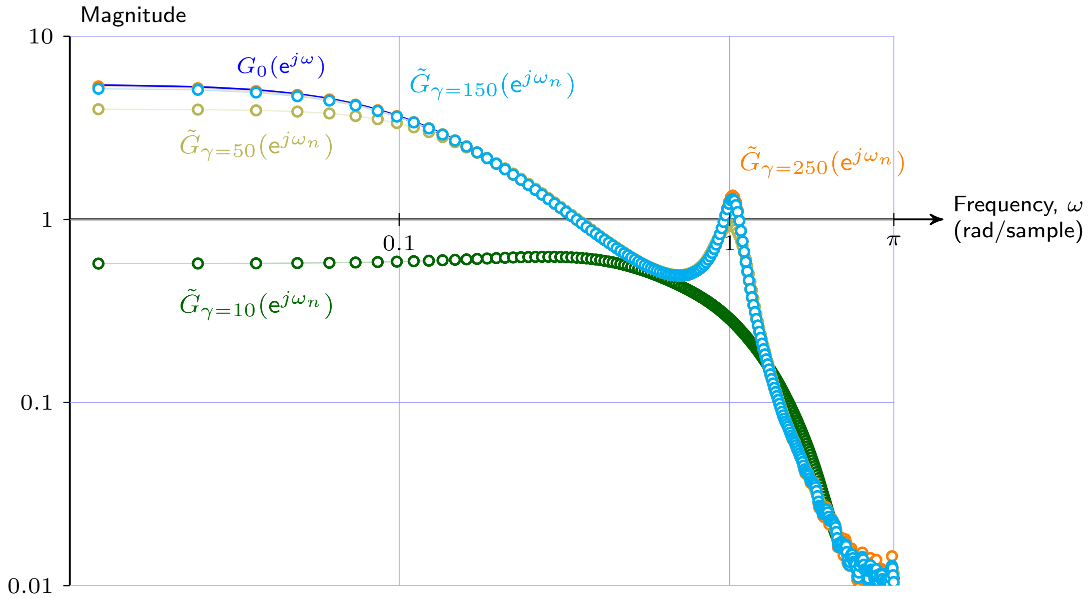
ETFE smoothing example

Mean-square errors versus γ



ETFE smoothing example

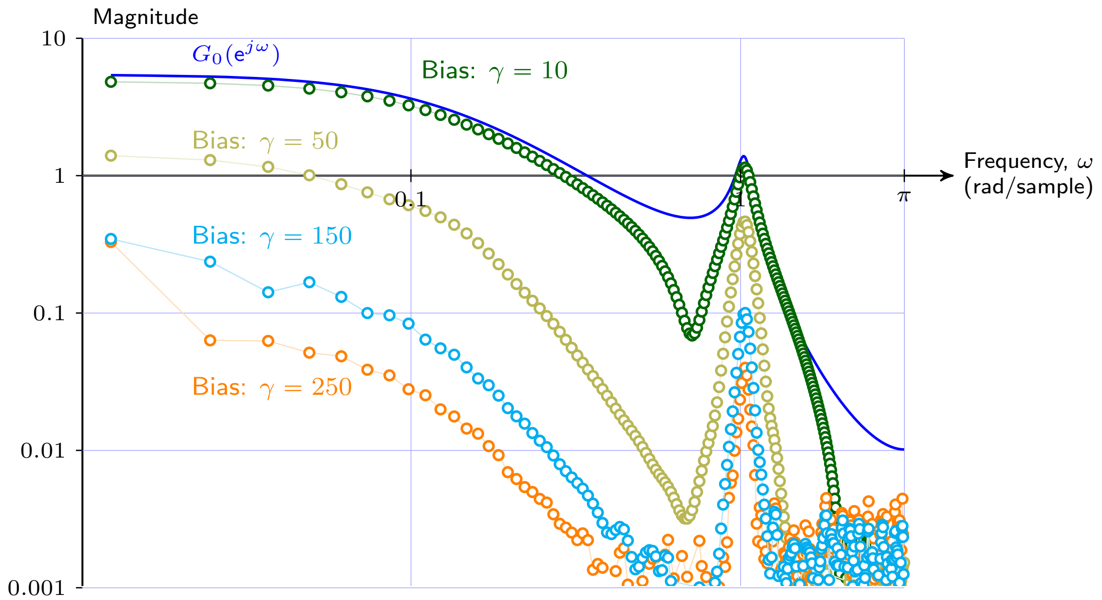
Expected smoothed estimate: $E\{\tilde{G}(e^{j\omega_n})\}$ (Estimated from 10,000 experiments)



ETFE smoothing example

Bias error: $E\left\{G_0(\mathrm{e}^{j\omega_n}) - E\left\{\tilde{G}_\gamma(\mathrm{e}^{j\omega_n})\right\}\right\}$

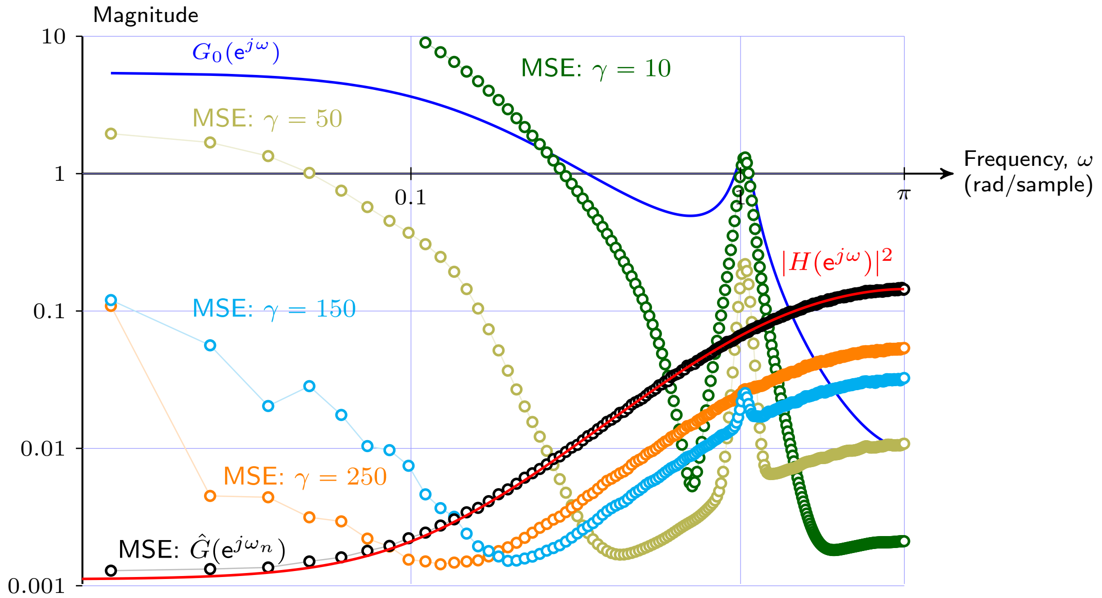
(Estimated from 10,000 experiments)



ETFE smoothing example

Mean-square error:

(Estimated from 10,000 experiments)
(Average of MSEs not MSE of averages)



ETFE smoothing example: MATLAB calculations

```
U = fft(u);                              % calculate N point FFTs
Y = fft(y);
Gest = Y./U;                             % ETFE estimate
Gs = 0*Gest;                             % smoothed estimate

[omega,Wg] = WfHann(gamma,N);            % window (centered)
zidx = find(omega==0);                   % shift to start at zero
omega = [omega(zidx:N);omega(1:zidx-1)]; % frequency grid
Wg = [Wg(zidx:N);Wg(1:zidx-1)];
a = U.*conj(U);                          % variance weighting

for wn = 1:N,
  Wnorm = 0;                             % reset normalisation
  for xi = 1:N,
    widx = mod(xi-wn,N)+1;               % wrap window index
    Gs(wn) = Gs(wn) + ...
           Wg(widx) * Gest(xi) * a(xi);
    Wnorm = Wnorm + Wg(widx) * a(xi);
  end
  Gs(wn) = Gs(wn)/Wnorm;                 % weight normalisation
end
```


Duality

Familiar relationship:

$$\underbrace{y(k) \;=\; \sum_{-\infty}^{\infty} g(i)u(i-k)}_{\text{Convolution in time domain}} \quad \longleftrightarrow \quad \underbrace{Y(\mathrm{e}^{j\omega_n}) = G(\mathrm{e}^{j\omega_n})U(\mathrm{e}^{j\omega_n})}_{\text{Multiplication in frequency domain}}$$

By duality, we also have:

$$\underbrace{\tilde{\Phi}(\mathrm{e}^{j\omega_n}) \;=\; \sum_{\xi=-N/2+1}^{N/2} W(\mathrm{e}^{j\xi})\Phi(\mathrm{e}^{j\omega_n - j\xi})}_{\text{Convolution}^1\text{ in frequency domain}} \quad \longleftrightarrow \quad \underbrace{\tilde{R}(\tau) \;=\; w(\tau)\,R_\tau}_{\text{Multiplication in time domain}}$$

[1] This is a circular convolution

Time-domain windows

Spectra

For periodic signals (with period $M = N$),

$$\frac{1}{N}\left|U_N(\mathrm{e}^{j\omega_n})\right|^2 = \phi_u(\mathrm{e}^{j\omega_n}) = \sum_{\tau=0}^{N-1} R_u(\tau)\mathrm{e}^{-j\omega_n\tau}$$
$$= \text{Fourier transform of } R_u(\tau).$$

Time-domain windows — smoothing the spectral estimates

Time-domain window

Define, via the inverse Fourier transform, a time-domain window,

$$w_\gamma(\tau) \;=\; \frac{1}{N}\sum_{n=-N/2+1}^{N/2} W_\gamma(\mathrm{e}^{j\omega_n})\mathrm{e}^{j\omega_n\tau}.$$

Time-domain windowing of spectral estimates

Then, the smoothed input spectral estimate, $\tilde{\phi}_u(\mathrm{e}^{j\omega_n})$, is

$$\sum_{\xi=-N/2+1}^{N/2} W_\gamma(\mathrm{e}^{j(\xi-\omega_n)})\frac{1}{N}\left|U_N(\mathrm{e}^{j\xi})\right|^2 \;\approx\; \sum_{\tau=-\infty}^{\infty} w_\gamma(\tau)\hat{R}_u(\tau)\mathrm{e}^{-j\tau\omega_n}$$

This is the Fourier transform of a time-domain multiplication.

Time-domain windows

Time-domain truncations

In practice choose $W_\gamma(\mathrm{e}^{j\omega})$ (or $w_\gamma(\tau)$) so that,

$$w_\gamma(\tau) = \begin{cases} 0 & \text{for} \quad \tau < -\gamma \\ > 0 & \quad -\gamma \leqslant \tau \leqslant \gamma \\ 0 & \quad \tau > \gamma \end{cases}$$

where we often have $\gamma \ll N$.

Windowed spectral estimate

Then, the smoothed input spectral estimate is

$$\tilde{\phi}_u(\mathrm{e}^{j\omega_n}) \;=\; \sum_{\tau=-\gamma}^{\gamma} w_\gamma(\tau)\hat{R}_u(\tau)\mathrm{e}^{-j\tau\omega_n},$$

and $\hat{R}_u(\tau)$ need only be calculated over $-\gamma \leqslant \tau \leqslant \gamma$.

Time-domain windows

Cross-spectral estimates

The smoothed cross-spectral estimate, $\tilde{\phi}_{yu}(\mathrm{e}^{j\omega_n})$, is

$$\begin{aligned}\tilde{\phi}_{yu}(\mathrm{e}^{j\omega_n}) \;&=\; \sum_{\xi=-N/2+2}^{N/2} W_\gamma(\mathrm{e}^{j(\xi-\omega_n)})\frac{1}{N}Y_N(\mathrm{e}^{j\xi})U_N^*(\mathrm{e}^{j\xi}) \\ &\approx \sum_{\tau=-\infty}^{\infty} w_\gamma(\tau)\hat{R}_{yu}(\tau)\mathrm{e}^{-j\tau\omega_n},\end{aligned}$$

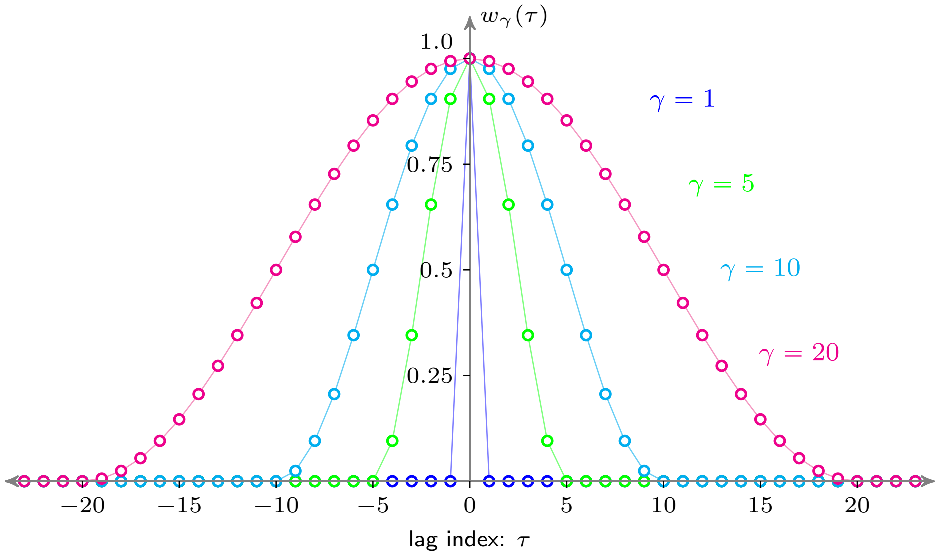
and in the finite support $w_\gamma(\tau)$ case,

$$= \sum_{\tau=-\gamma}^{\gamma} w_\gamma(\tau)\hat{R}_{yu}(\tau)\mathrm{e}^{-j\tau\omega_n}$$

Again, $\hat{R}_{yu}(\tau)$ need only be calculated over $-\gamma \leqslant \tau \leqslant \gamma$.

Time-domain windows

Hann windows in the time-domain (scaled to $w_\gamma(0) = 1$)



$$w_\gamma(\tau) = 0, \qquad \text{for } |\tau| > \gamma.$$

Window functions

Bartlett window

$$W_\gamma(\mathrm{e}^{j\omega}) = \frac{1}{\gamma}\left(\frac{\sin\gamma\omega/2}{\sin\omega/2}\right)^2 \qquad w_\gamma(\tau) = 1 - \frac{|\tau|}{\gamma}, \quad 0 \leqslant |\tau| \leqslant \gamma.$$

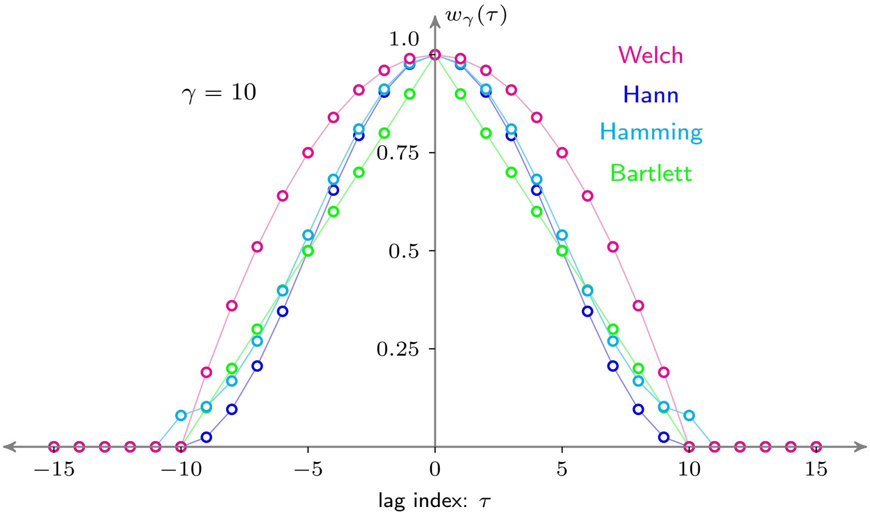
Hann window

$$W_\gamma(\mathrm{e}^{j\omega}) = \frac{1}{2}D_\gamma(\omega) + \frac{1}{4}D_\gamma(\omega - \pi/\gamma) + \frac{1}{4}D_\gamma(\omega + \pi/\gamma) \qquad w_\gamma(\tau) = \frac{1}{2}\left(1 + \cos\frac{\pi\tau}{\gamma}\right)$$

where $D_\gamma(\omega) = \dfrac{\sin\omega(\gamma + 0.5)}{\sin\omega/2}$

Common windows

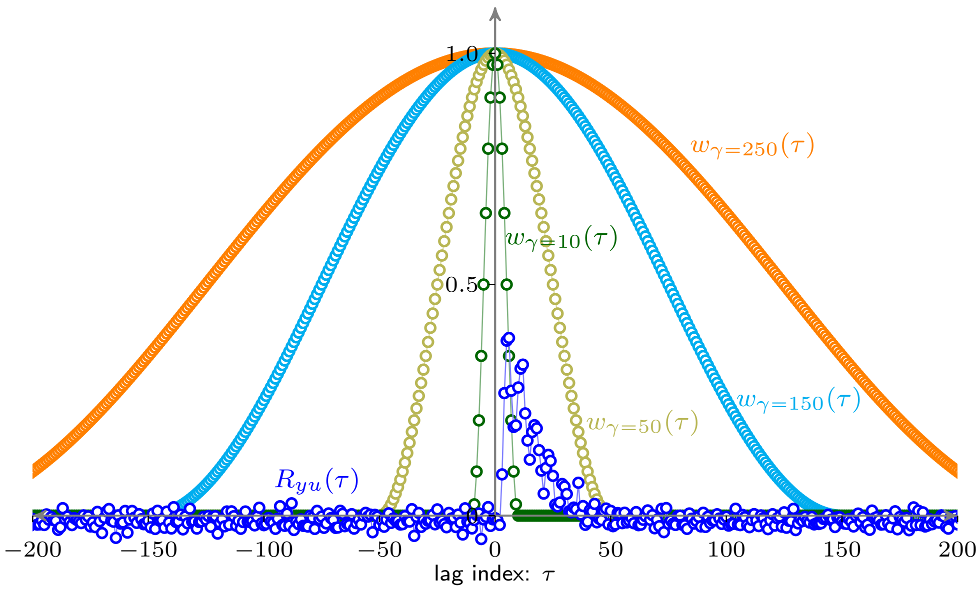
Time-domain representations



$$w_\gamma(\tau) = 0, \qquad \text{for } |\tau| > \gamma.$$

ETFE smoothing example

$R_{yu}(\tau)$ calculation and time-domain windows, $w_\gamma(\tau)$:



Time-domain windows

Window characteristics: width (specified by γ parameter)

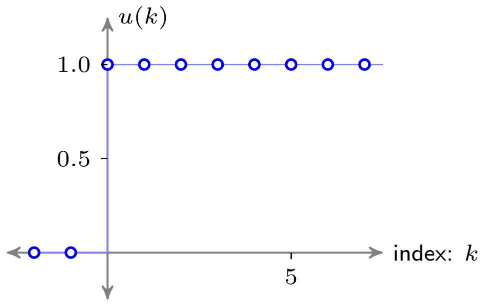
Decreasing γ: narrower $w_\gamma(\tau)$; wider $W_\gamma(\mathrm{e}^{j\omega})$

- the more frequencies, $\hat{G}(\mathrm{e}^{j\omega_n})$, included in the smoothing,
- the fewer $\hat{R}(\tau)$ estimates included in the smoothing,
- the smoother the result,
- the lower the noise induced variance,
- the higher the bias.

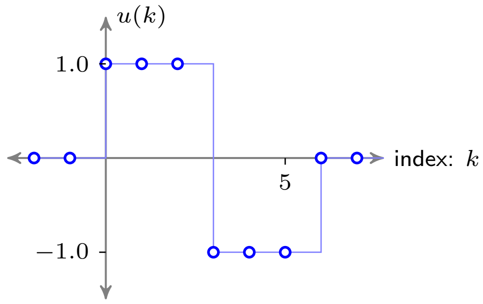
Input signals

Typical Operating signals

Steps



Doublets



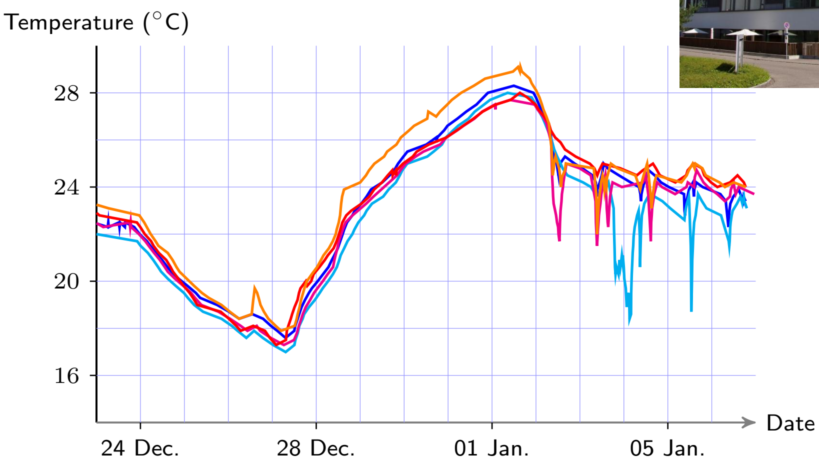
Sinusoids

Swept sinusoids: R experiments: $u_n = \sin \omega_n k, \quad n = 1, \ldots, R.$

Variants: chirp signals; multi-sines (Schroeder phasing).

Doublet excitation

Building identification experiments



Input signals

Filtered white noise

$$u = L(z)e, \quad e \in \mathcal{N}(0,1) \qquad \phi_u(\mathrm{e}^{j\omega}) \; = \; \left| L(\mathrm{e}^{j\omega}) \right|^2 .$$

Designing $L(z)$ specifies (in expectation) the frequency content of the input signal.

Random binary signal

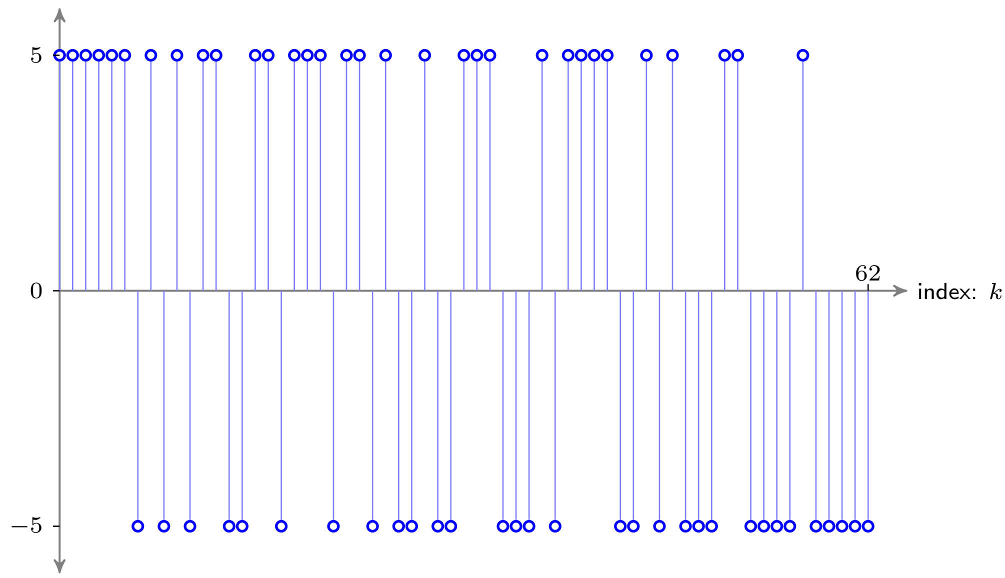
$$u(k) \; = \; a \; \mathrm{sign}(e(k)), \quad e \in \mathcal{N}(0,1), \qquad R_u(\tau) \; = \; \begin{cases} a^2, & \tau = 0 \\ 0 & \tau \neq 0, \end{cases} \qquad \phi_u(\mathrm{e}^{j\omega}) \; = \; a^2 .$$

Weighted version: $\quad u(k) \; = \; a \; \mathrm{sign}(L(z)e(k)), \quad e \in \mathcal{N}(0,1)$

$L(z)$ is used to modify the spectrum.

PRBS input signals

Input sequence ($X = 6,\ M = 63$)



PRBS signals

Run length distribution

1/2 runs of length 1
1/4 runs of length 2
1/8 runs of length 3
⋮

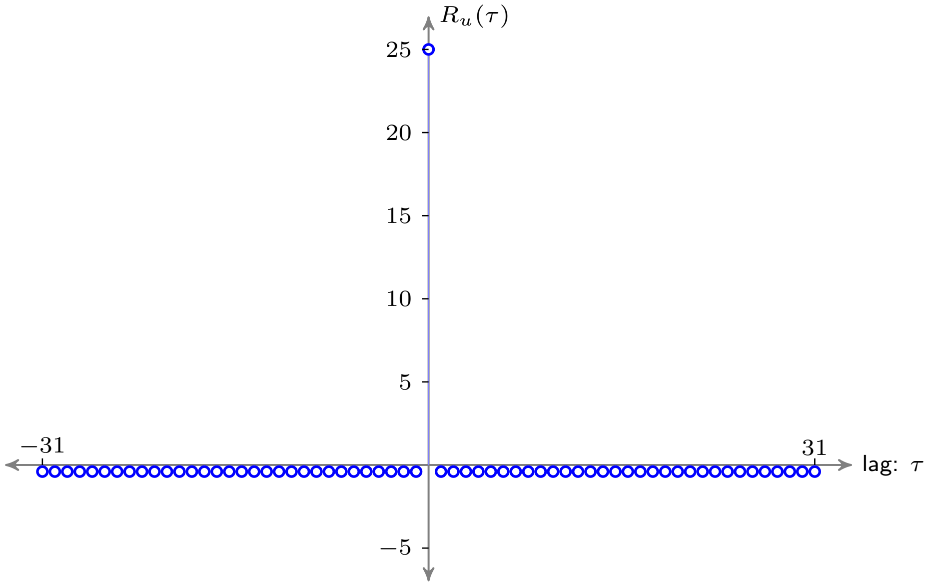
Autocorrelation function

Using a calculation length of 1 period: $N = M = 2^X - 1$.

$$R_u(\tau) = \frac{1}{N}\sum_{k=0}^{N-1} u(k)u(k-\tau) = \begin{cases} a^2 & \text{if } \tau = 0, \\ \dfrac{-a^2}{(2^X - 1)} & \text{if } \tau \neq 0 \end{cases}$$

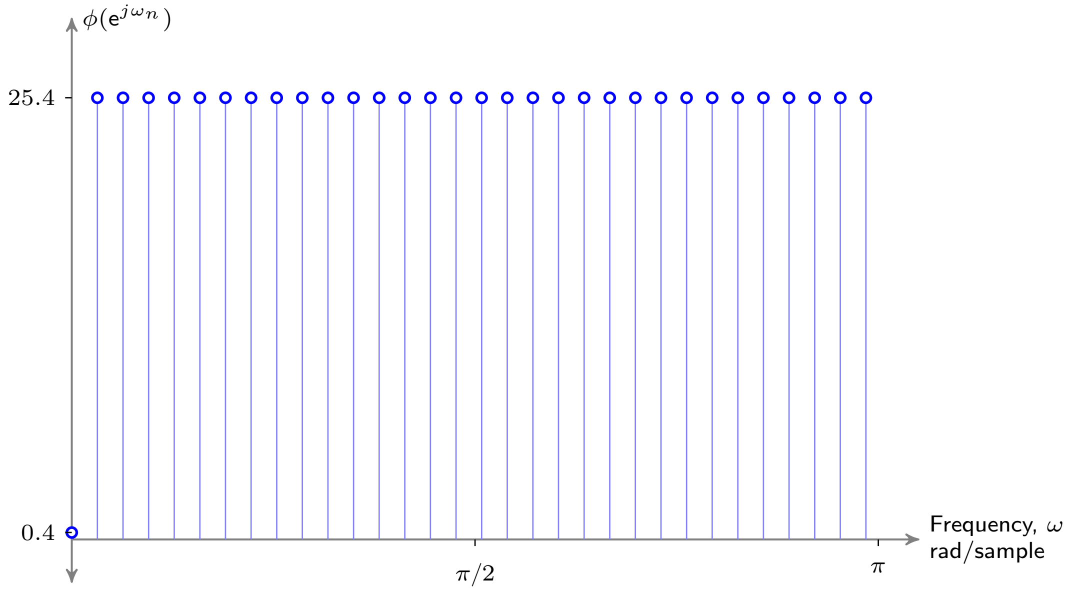
PRBS input signals

Autocorrelation: $R_u(\tau)$ (calculation with period $N = M = 2^X = 63$)



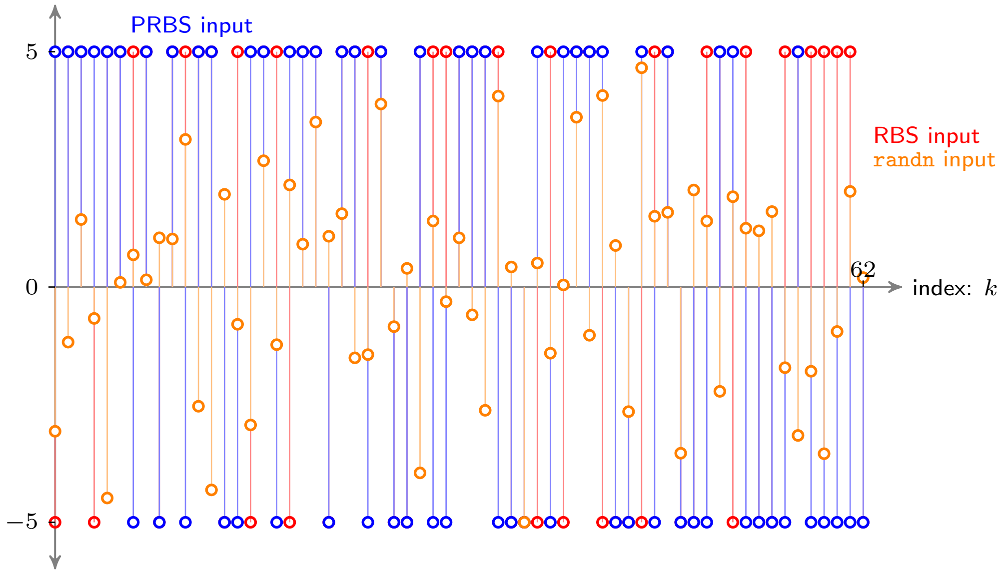
PRBS input signals

Input signal spectra: $\phi_u(e^{j\omega_n})$



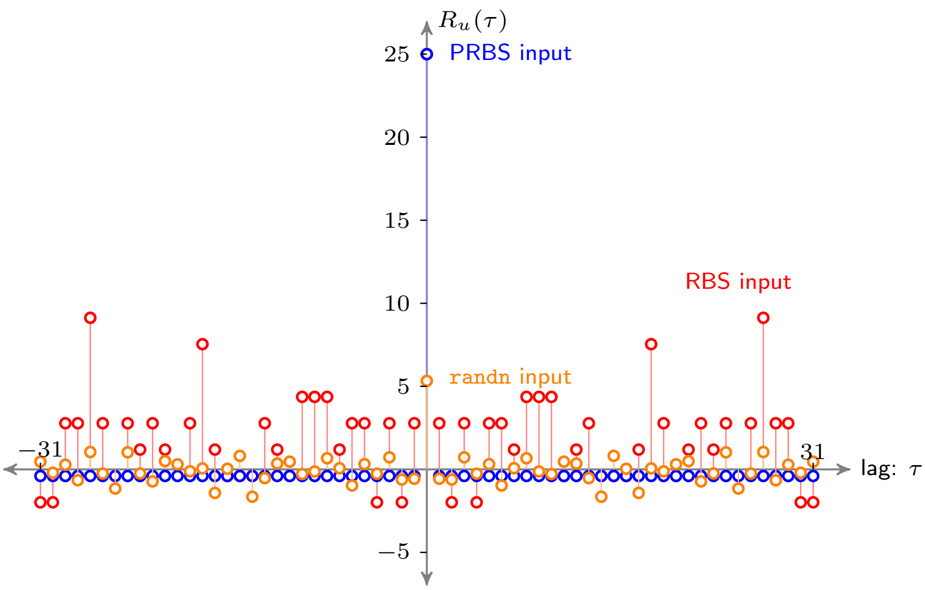
Random input signals (periodic)

Random (normally distributed) signal, scaled to ± 5.



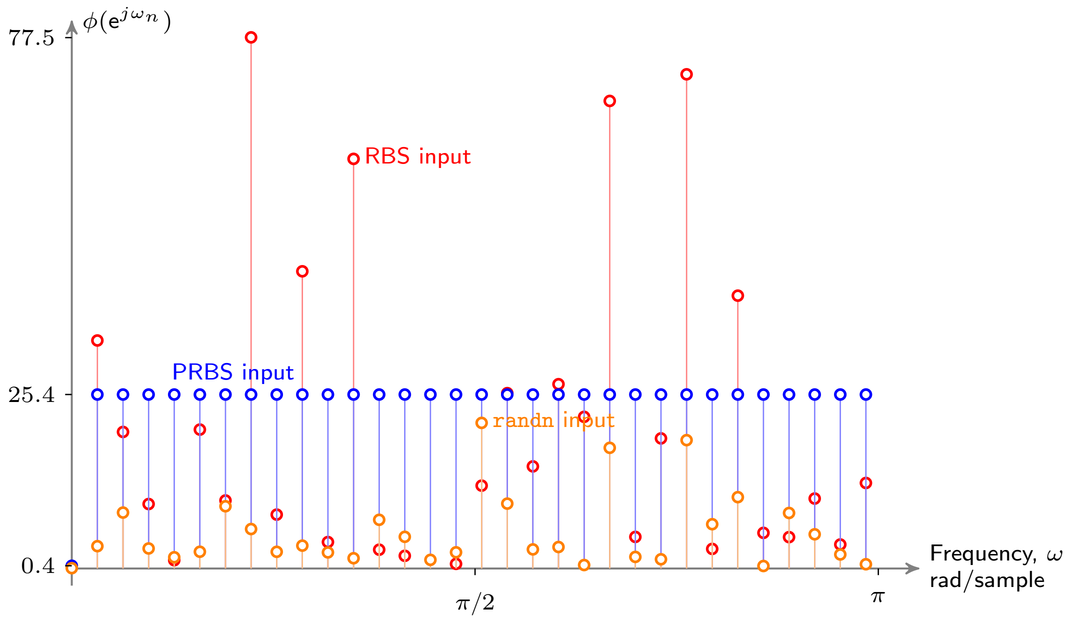
Random input signals (periodic)

Periodic autocorrelation: $R_u(\tau) \;=\; \dfrac{1}{N}\displaystyle\sum_{k=0}^{N-1} u(k)u(k-\tau)$



Input signals

Input signal spectra, $\phi_u(e^{j\omega_n})$: PRBS, `randn` and RBS signals.



Multi-sinusoidal signals

Sum of (harmonically related) sinusoids

$$u(k) = \sum_{s=1}^{S} \sqrt{2\alpha_s} \cos(\omega_s kT + \phi_s) \quad \text{where } T \text{ is the sampling period,}$$

Select S harmonically related frequencies:

$$\omega_s = \frac{2\pi l s}{N}, \quad s = 1, \ldots, S.$$

Lowest frequency is $\dfrac{2\pi l}{N}$ where l is an integer.

Highest frequency must be less than the Nyquist frequency:

$$\frac{2\pi l S}{N} < \pi, \qquad \Longrightarrow \qquad S < \frac{N}{2l}.$$

$$\text{Total signal power} = \sum_{s=1}^{S} \alpha_s = 1 \qquad \text{(normalised).}$$

Multi-sinusoidal signals

Schroeder phasing

$$u(k) = \sum_{s=1}^{S} \sqrt{2\alpha_s} \cos\left(\omega_s kT + \phi_s\right)$$

Select the phases, ϕ_s, to minimise the peak amplitude of $u(k)$.

Solution: $\qquad \phi_s = 2\pi \sum_{n=1}^{s} n\alpha_s.$

For equal power in each sinusoid:

$$\alpha_s = 1/S \qquad \text{and} \qquad \phi_s = \frac{\pi(s^2+s)}{S}.$$

General case: arbitrary selection of frequencies

Consider $S = N/2$ sinusoids ($l = 1$).

$\alpha_s > 0$ for $s \in$ {selected frequencies set}, $\qquad \alpha_s = 0$ otherwise.

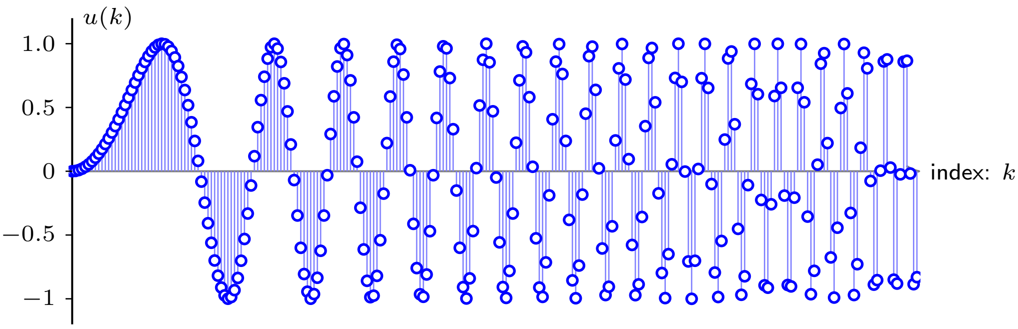
Chirp signals

Lowest frequency: $\dfrac{2\pi k_1}{K}$, $\qquad$ Highest frequency: $\dfrac{2\pi k_2}{K}$ $\qquad$ [radians/sample]

Define: $\quad a = \dfrac{\pi(k_2 - k_1)}{K^2}, \qquad b = \dfrac{2\pi k_1}{K^2},$

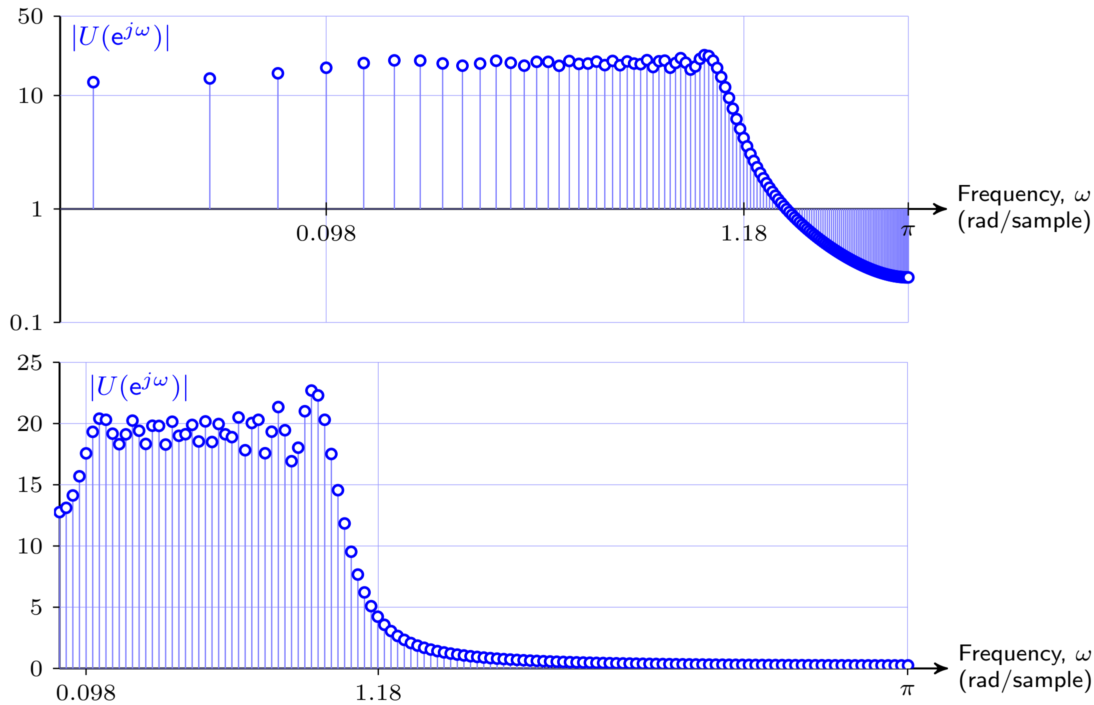
Input signal:

$$u(k) = \sin\left((ak+b)k\right), \quad k = 0, 1, \ldots, K-1.$$



$k_1 = 4$
$k_2 = 48$
$\omega_{\text{lowest}} = 0.098$
$\omega_{\text{highest}} = 1.18$

Chirp signals



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