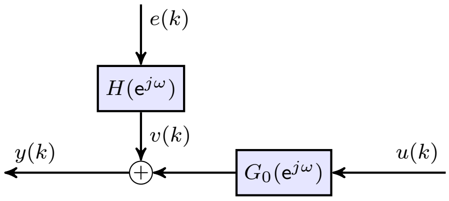
System Identification

Supplementary notes: lecture 5

Roy Smith

5 Frequency-domain identification

5.1 Input-output relationships: finite-energy signals



$$y(k) \ = \ \sum_{l=0}^{\infty} g(l)u(k-l) \ + \ v(k),$$

$$Y(\mathrm{e}^{j\omega}) \ = \ G(\mathrm{e}^{j\omega})U(\mathrm{e}^{j\omega}) + V(\mathrm{e}^{j\omega})$$

This serves as a motivation for the estimate of $G(\mathrm{e}^{j\omega})$ as the ratio of Fourier Transform estimates.

$$\begin{aligned} Y(\mathrm{e}^{j\omega}) \ &= \ \sum_{k=-\infty}^{\infty} y(k)\mathrm{e}^{-j\omega k} \ = \ \sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} g(l)u(k-l)\mathrm{e}^{-j\omega k} \ + \ \sum_{k=-\infty}^{\infty} v(k)\mathrm{e}^{-j\omega k} \\ &= \ \sum_{l=0}^{\infty} g(l)\mathrm{e}^{-j\omega l} \sum_{k=-\infty}^{\infty} u(k-l)\mathrm{e}^{-j\omega(k-l)} \ + \ V(\mathrm{e}^{j\omega}), \\ &= \ \sum_{l=0}^{\infty} g(l)\mathrm{e}^{-j\omega l} \sum_{m=-\infty}^{\infty} u(m)\mathrm{e}^{-j\omega m} \ + \ V(\mathrm{e}^{j\omega}) \quad (m \ = \ k-l) \\ &= \ G(\mathrm{e}^{j\omega})\,U(\mathrm{e}^{j\omega}) \ + \ V(\mathrm{e}^{j\omega}). \end{aligned}$$

The result holds true for the Fourier Transforms in the expected value case:

$$E\{v(k)\} = 0 \implies E\{V(\mathrm{e}^{j\omega})\} = 0 \quad \text{(via linearity of expectation operator)}$$

So,

$$E\{Y(\mathrm{e}^{j\omega})\} = G(\mathrm{e}^{j\omega})E\{U(\mathrm{e}^{j\omega})\}.$$

This is "idealised" in several senses:

1. The Fourier transform of $v(k)$ is assumed to exist. As noise is typically finite power, and not finite energy, this won't be satisfied. However we are more interested in the relationship between $U(\mathrm{e}^{j\omega})$ and $Y(\mathrm{e}^{j\omega})$ and so we will overlook this issue for the time being.
2. We can only estimate $Y(\mathrm{e}^{j\omega})$ and $U(\mathrm{e}^{j\omega})$. In practice we will only have finite data and so the infinite summations above can usually[1] only be approximated by finite truncations.
3. Even if $u(k)$ has a non-zero value on a finite range of values of k, the output, $y(k)$, will not share this property unless the impulse response, $g(l)$, also has finite support.

5.2 ETFE for periodic signals

In the periodic signal case it turns out that an exact input-output frequency domain relationship can be calculated from finite data.

The time domain response of the system is,

$$\begin{aligned} y(k) &= G(z)\,u(k) + v(k), \qquad k = 0, \ldots, N-1, \\ &= \sum_{i=0}^{\infty} g(i)u(k-i) + v(k) \\ &= \sum_{i=-\infty}^{\infty} g(i)u(k-i) + v(k). \end{aligned}$$

[1]Periodic signals are an exception to this as only one period is needed to define the entire signal.

By taking the DFT of $y(k)$ we get,

$$
\begin{aligned}
Y_N(\mathrm{e}^{j\omega_n}) &= \sum_{k=0}^{N-1} y(k)\mathrm{e}^{-j\omega_n k} \\
&= \sum_{k=0}^{N-1} \sum_{i=-\infty}^{\infty} g(i)u(k-i)\mathrm{e}^{-j\omega_n k} + \sum_{k=0}^{N-1} v(k)\mathrm{e}^{-j\omega_n k} \\
&= \underbrace{\sum_{i=-\infty}^{\infty} g(i)\mathrm{e}^{-j\omega_n i}}_{G(\mathrm{e}^{j\omega_n})} \sum_{k=0}^{N-1} u(k-i)\mathrm{e}^{-j\omega_n (k-i)} + V_N(\mathrm{e}^{j\omega_n}) \\
&= G(\mathrm{e}^{j\omega_n}) \sum_{l=-i}^{N-1-i} u(l)\mathrm{e}^{-j\omega_n l} + V_N(\mathrm{e}^{j\omega_n}) \\
&= G(\mathrm{e}^{j\omega_n}) \underbrace{\left(\sum_{l=0}^{N-i-1} u(l)\mathrm{e}^{-j\omega_n l} + \sum_{l=N-i}^{N-1} u(l+N)\mathrm{e}^{-j\omega_n (l+N)} \right)}_{U_N(\mathrm{e}^{j\omega_n})} + V_N(\mathrm{e}^{j\omega_n})
\end{aligned}
$$

This argument assumes $l < N$. If this is not satisfied then shift the index by an appropriate multiple of N.

Note that the true plant, $G(\mathrm{e}^{j\omega_n})$, appears in the relationship. So $G(\mathrm{e}^{j\omega_n})$ is given by,

$$
\frac{Y_N(\mathrm{e}^{j\omega_n})}{U_N(\mathrm{e}^{j\omega_n})} \; = \; G(\mathrm{e}^{j\omega_n}) \; + \; \frac{V_N(\mathrm{e}^{j\omega_n})}{U_N(\mathrm{e}^{j\omega_n})}.
$$

For periodic signals then, the ETFE gives an unbiased estimate of $G(\mathrm{e}^{j\omega_n})$. There is no error arising from the truncation of finite data as periodic signals are completely defined by the data on a single period.

Note however that noise is still present and will give a variance error. The summation to $-\infty$ makes it clear that we need $u(k)$ and $y(k)$ to be periodic in negative time as well. In practice this is never exactly achieved.

5.3 Transients in ETFE methods

The average magnitude of $U_N(\mathrm{e}^{j\omega_n})$ grows with a rate of N (periodic case) or $\sqrt{N}$ (random case). As we are dividing by $U_N(\mathrm{e}^{j\omega_n})$ the transient decays with a rate of $1/N$ in the periodic input case, or $1/\sqrt{N}$ in the random input case.

To see this note that the total energy in the time and frequency domains is the same. Consider the periodic and the random cases.

Periodic $u(k)$ case:

The DFT for a single length M period of a signal is,

$$U_M(\mathrm{e}^{j\omega_n}) = \sum_{k=0}^{M-1} u(k)\mathrm{e}^{jk\omega_n}, \quad m = n, \ldots, M.$$

Now consider increasing the experiment length to $N = mM$ where m is an integer. This adds $m-1$ more periods to the experiment. Then,

$$\begin{aligned} U_N(\mathrm{e}^{j\omega_n}) = U_{mM}(\mathrm{e}^{j\omega_n}) &= \sum_{k=0}^{mN-1} u(k)\mathrm{e}^{jk\omega_n} = \sum_{r=1}^{m}\sum_{s=0}^{M-1} u(s+rM)\mathrm{e}^{j(s+rM)\omega_n} \\ &= \sum_{r=1}^{m}\sum_{s=0}^{M-1} u(s)\mathrm{e}^{js\omega_n} \underbrace{\mathrm{e}^{jrM\omega_n}}_{\substack{=1 \\ \text{as } M\omega_n = 2\pi}} = m\sum_{s=0}^{M-1} u(s)\mathrm{e}^{js\omega_n} = mU_N(\mathrm{e}^{j\omega_n}) \end{aligned}$$

So we see that the magnitude of the DFT frequencies are scaled by m. It is important to note that there are still only M frequencies that can be calculated. The energy at the other frequencies is zero.

This implies that, for periodic $u(k)$,

$$\lim_{m\longrightarrow\infty} \frac{R_{mM}(\mathrm{e}^{j\omega_n})}{U_M(\mathrm{e}^{j\omega_n})} = 0 \quad \text{with rate } 1/m.$$

Random $u(k)$ case:

This result for random $u(k)$ can be seen from the proof of convergence of the periodogram to the spectrum[2]. From this proof we have,

$$\begin{aligned} E\left\{\left|U_N(\mathrm{e}^{j\omega_n})\right|^2\right\} &= E\left\{U_N(\mathrm{e}^{j\omega_n})U_N(\mathrm{e}^{-j\omega_n})\right\} \\ &= N\phi_u(\mathrm{e}^{j\omega_n}) + 2c, \end{aligned}$$

where

$$|c| \leqslant C = \sum_{\tau=1}^{\infty} \left|\tau\, R_u(\tau)\right|.$$

As the power spectral density of $u(k)$ is constant, this immediately gives the random result.

References

[1] L. Ljung, *System Identification: Theory for the User*, 2nd ed. Prentice-Hall, 1999.

[2] See Ljung [1] for details.