System Identification

Lecture 4: Sampled dynamic models & frequency-domain analysis

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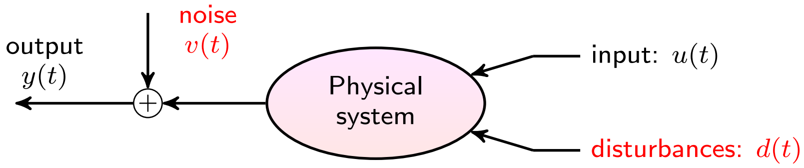
Continuous dynamics

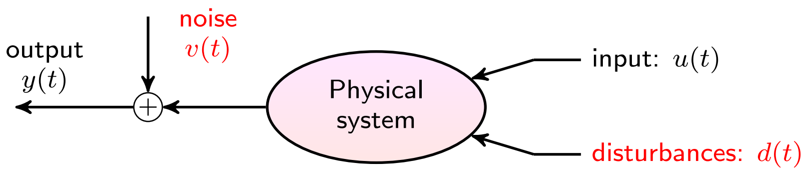
Framework

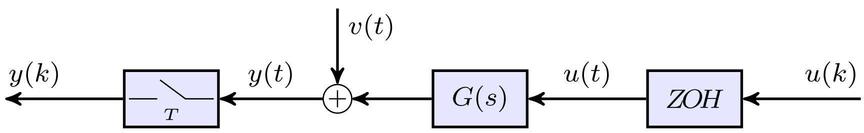


Linear time invariant system

Sampled linear dynamical system

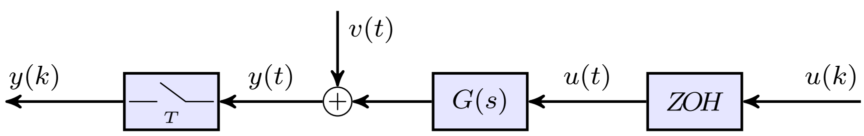
Framework





Sampled-data configuration

A typical practical configuration

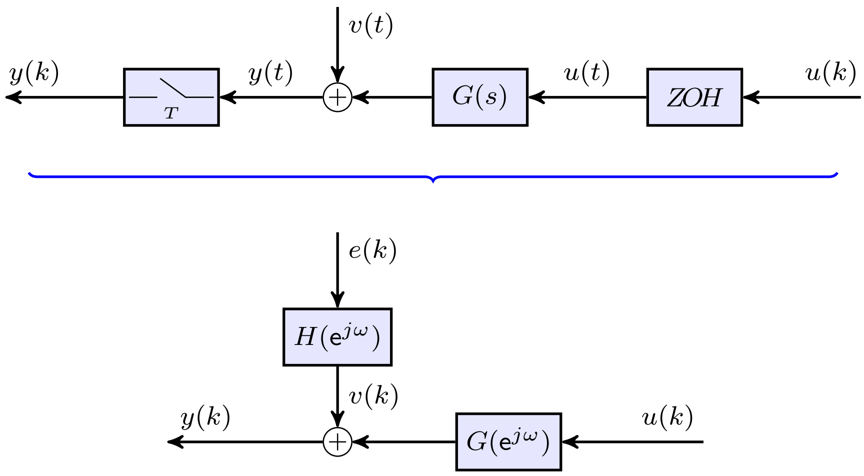


ZOH block:	Digital-to-analog converter (DAC)
sampling (T) block:	Analog-to-digital convertor (ADC)

The selection of the sampling time, T, will determine the maximum frequency that we can model with a discrete-time system.

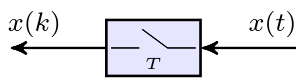
Sampled-data configuration

An equivalent model:

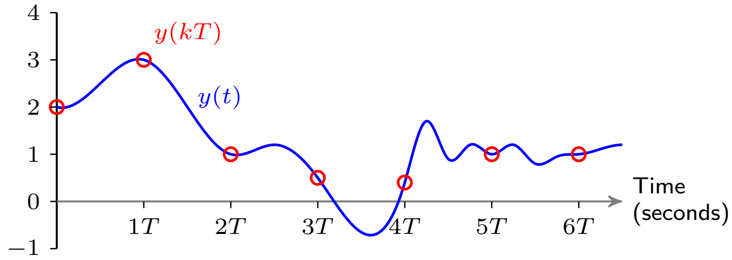


Sampling operation

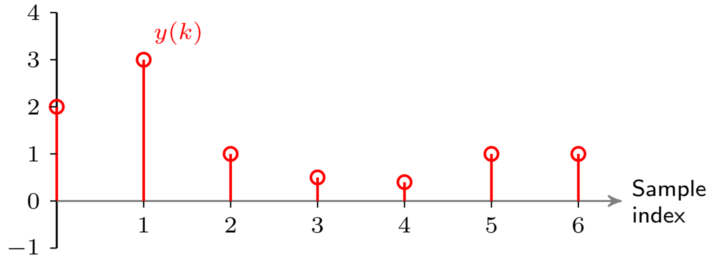
$$y(k) = y(t)\Big|_{t=kT,k=0,1,2,\dots}$$



Continuous signal:



Discrete sequence:



Discrete frequencies: Fourier series of periodic signals

Periodic signal: $x(k)$ (period $= M$ samples, assume M is even).

The Fourier series is:

$$X(\mathrm{e}^{j\omega_m}) = \sum_{k=0}^{M-1} x(k)\mathrm{e}^{-j\omega_m k}, \quad \text{where} \quad \omega_m = \frac{2\pi m}{M} = m\omega_1, \quad \omega_1 = \frac{2\pi}{M}$$

$$m = 0, \ldots, M-1.$$

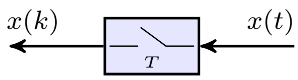
This is also periodic (with period $= M$).

Non-negative frequencies are: $m = 0$ to $m = M/2$,

corresponding to: $\omega_m = 0, \dfrac{2\pi}{M}, \dfrac{4\pi}{M}, \ldots, \dfrac{2\pi(M/2-1)}{M}, \pi.$

Sampled periodic signals

Sampling operation:



Periodic signal with period $= \tau_p$,

$$x(t) = x(t+\tau_p), \quad t \in (-\infty, \infty).$$

Now sample $x(t)$ with sampling time: T

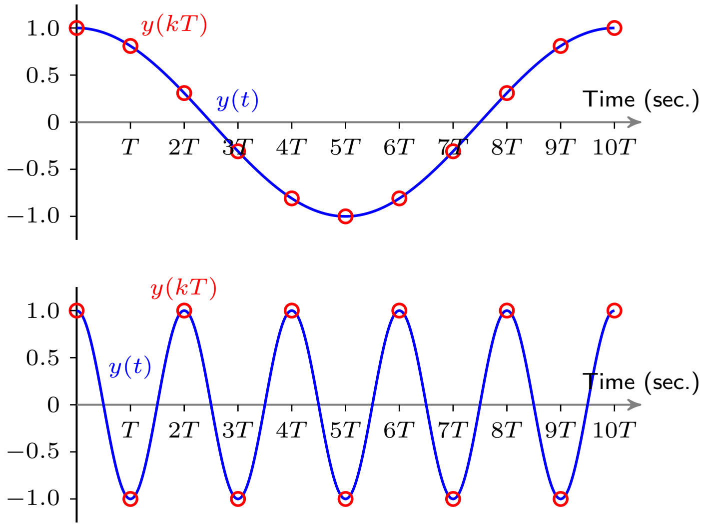
To get M samples per period (i.e. $MT = \tau_p$),

$$T = \frac{\tau_p}{M}.$$

This gives frequencies,

$$0, \underbrace{\frac{2\pi}{\tau_p}}_{\text{Fundamental frequency}}, \underbrace{2\left(\frac{2\pi}{\tau_p}\right), 3\left(\frac{2\pi}{\tau_p}\right), \cdots, \frac{M}{2}\left(\frac{2\pi}{\tau_p}\right)}_{\text{Harmonics}}$$

Sampled periodic signals



Lowest frequency:

$$\omega_u = \omega_1 = \frac{2\pi}{MT}$$

(apart from zero)

$$M = 10$$

Highest frequency:

$$\omega_u = \omega_{M/2} = \frac{\pi}{T}$$

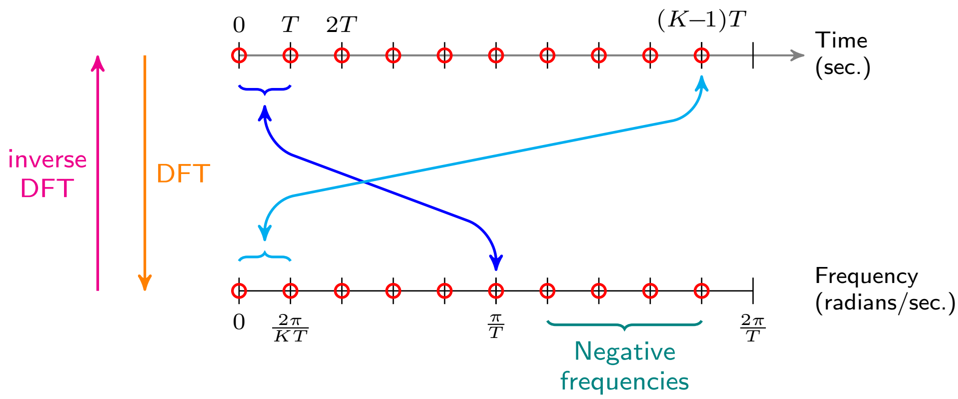
(Nyquist frequency)

There are only $M/2 + 1$ distinct frequencies (counting zero) that can be represented in a sampled signal.

Time and frequency domain sampling relationships

The sampling time, T, determines the Nyquist frequency π/T.

The experiment length, KT, determines the frequency spacing, $2\pi/KT$.



Parametric models

Model

$$y = G(\theta)\,u + H(\theta)\,e, \qquad y \in \mathcal{R}, u \in \mathcal{R}, \theta \in \mathcal{R}^d \qquad (d \text{ may be infinite}).$$

Key features

Model — Experiment relationship

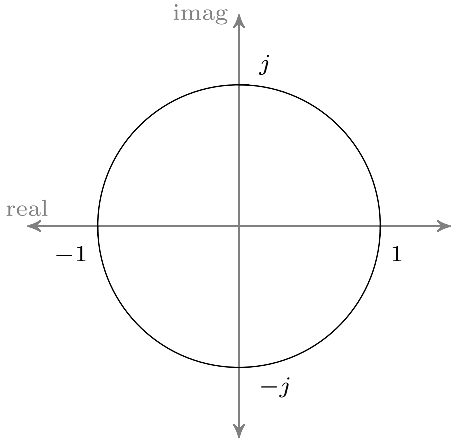
Finite experiments

Define the measurement data set:

$$Z_K = \{u(0), y(0), \ldots, u(K-1), y(K-1)\}.$$

Unmeasured past

Frequency domain parametrisation



Time-domain parametrisations

Pulse response

$$y(k) = \sum_{i=0}^{\infty} g(i)u(k-i)$$

Real-rational transfer functions

$$G = \frac{B(z)}{A(z)} = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \dots b_m z^{n-m}}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots a_n}$$
$$= \frac{b_1 z^{-1} + b_2 z^{-2} + \dots b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots a_n z^{-n}}$$

State-space representations

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

Parametrised model sets

We are looking for an estimate, $\hat{G}$, in a parametrised set, $\{G(\theta)\}$,

that most closely matches the experimental data.

$\theta \in \mathcal{R}^d$ is the parameter vector.

	Model structure	Parameter vector, $\theta \in \mathcal{R}^d$
Frequency response:	$G(e^{j\omega})$	$G(e^{j\omega_n}), \quad n = 0, \dots, N-1.$
Pulse response:	$g(k)$	$\begin{bmatrix} g(0) & g(1) & \dots \end{bmatrix}^T$
Transfer function:	$\dfrac{B(z)}{A(z)}$	$\begin{bmatrix} a_1 & \dots & b_1 & \dots \end{bmatrix}^T$
State-space:	$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	$\begin{bmatrix} A_{ij} & \dots & B_{ij} & \dots & C_{ij} & \dots \end{bmatrix}^T$

Discrete periodic signals

If $x(k)$ is periodic with period equal to M (assume M even);

$$x(k) = x(k+M), \quad \text{for all } k \in \{-\infty, \infty\}.$$

The fundamental frequency is,

$$\omega_1 = \frac{2\pi}{M}.$$

There are only M unique harmonics of the sinusoid, $e^{j\omega_1}$.

The non-negative harmonic frequencies are,

$$e^{jn\omega_1}, \quad n = 0, 1, \dots, M/2.$$

Discrete Fourier series (periodic signals)

Periodic signal: $x(k)$ (period $= M$).

Choose the "calculation length", N to be equal to the period ($N = M$).

The Fourier series is:

$$X(e^{j\omega_n}) = \sum_{k=0}^{N-1} x(k)e^{-j\omega_n k}, \quad \text{where} \quad \omega_n = \frac{2\pi n}{N} = n\omega_1, \quad n = 0, \ldots, N-1.$$

The inverse transform is:

$$x(k) = \frac{1}{N}\sum_{n=0}^{N-1} X(e^{j\omega_n})e^{j\omega_n k}.$$

Autocorrelation (periodic signals, $N = M$)

The autocorrelation of $x(k)$ (of period M and $N = M$) is:

$$R_x(\tau) = \frac{1}{N}\sum_{k=0}^{N-1} x(k)x(k-\tau).$$

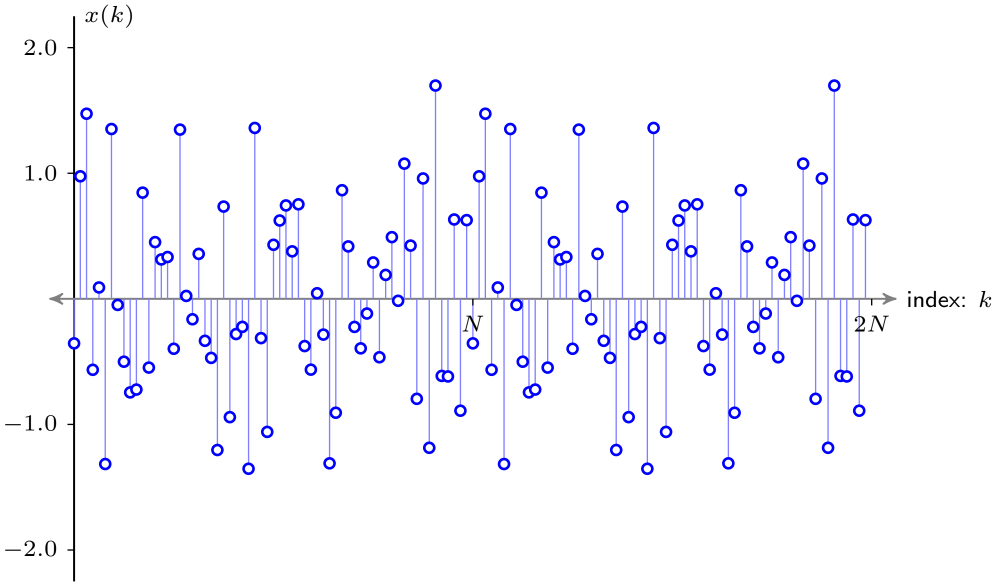
The Fourier transform of $R_x(\tau)$ is defined as the power spectral density,

$$\phi_x(e^{j\omega_n}) = \sum_{\tau=0}^{N-1} R_x(\tau)e^{-j\omega_n \tau} = \frac{1}{N}|X(e^{j\omega_n})|^2$$

Energy in a single period:

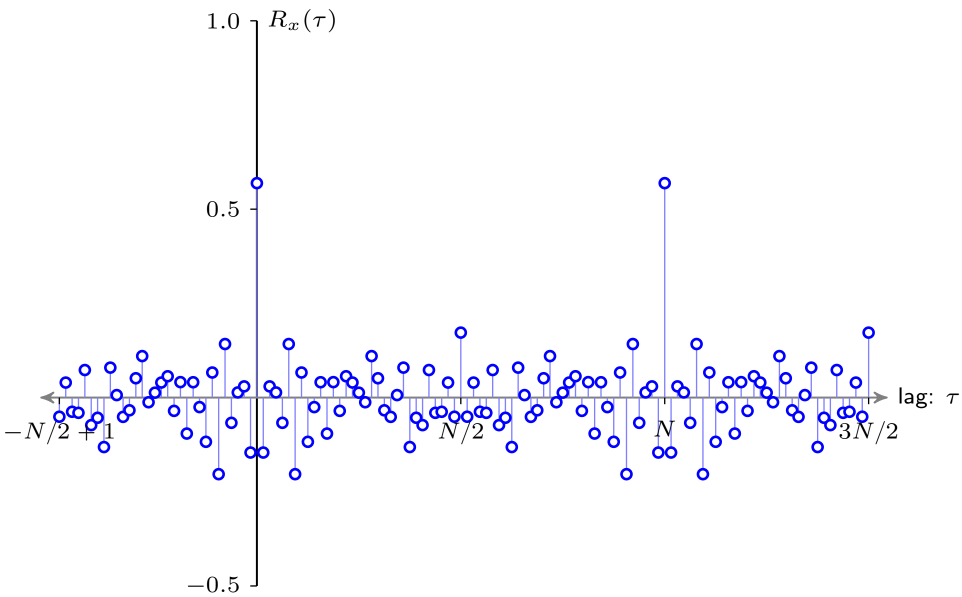
$$\sum_{k=0}^{N-1} |x(k)|^2 = \sum_{n=0}^{N-1} \phi_x(e^{j\omega_n})$$

Example: periodic signal ($N = M$)



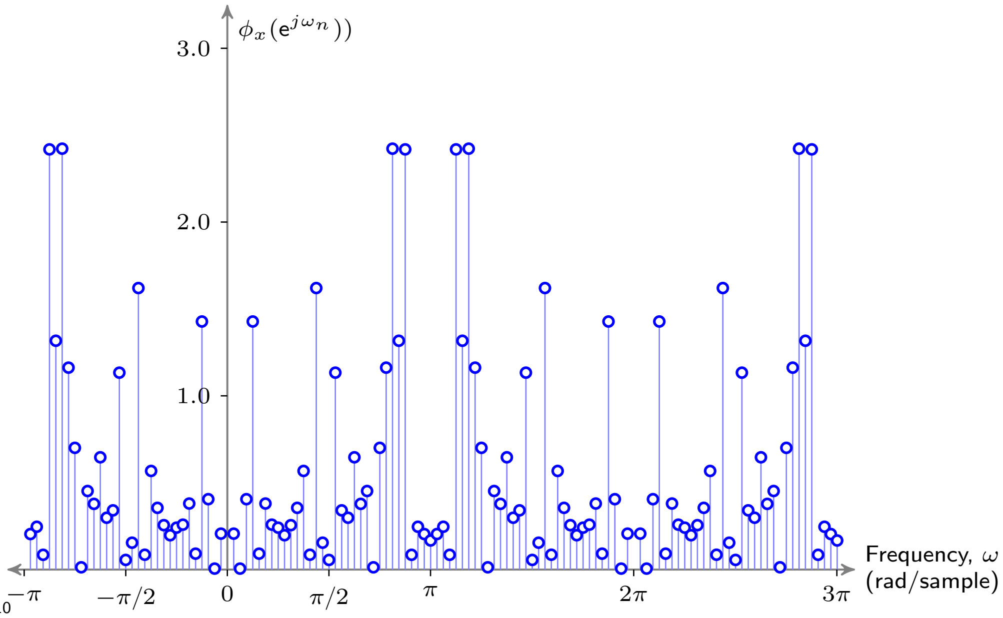
Autocorrelation example (periodic signal, $N = M$)

$$R_x(\tau) = \frac{1}{N} \sum_{k=0}^{N-1} x(k)x(k-\tau), \qquad \tau = -N/2+1, \ldots, N/2.$$



Power spectral density (periodic signal, $N = M$)

$$\phi_x(\mathrm{e}^{j\omega_n}) = \sum_{\tau=0}^{N-1} R_x(\tau)\mathrm{e}^{-j\omega_n\tau}, \quad \omega_n = \frac{2\pi n}{N}, \quad n = -N/2+1, \ldots, N/2.$$



Cross-correlation (periodic signals, $N = M$)

The cross-correlation of $y(k)$ and $u(k)$ (both of period $M = N$) is:

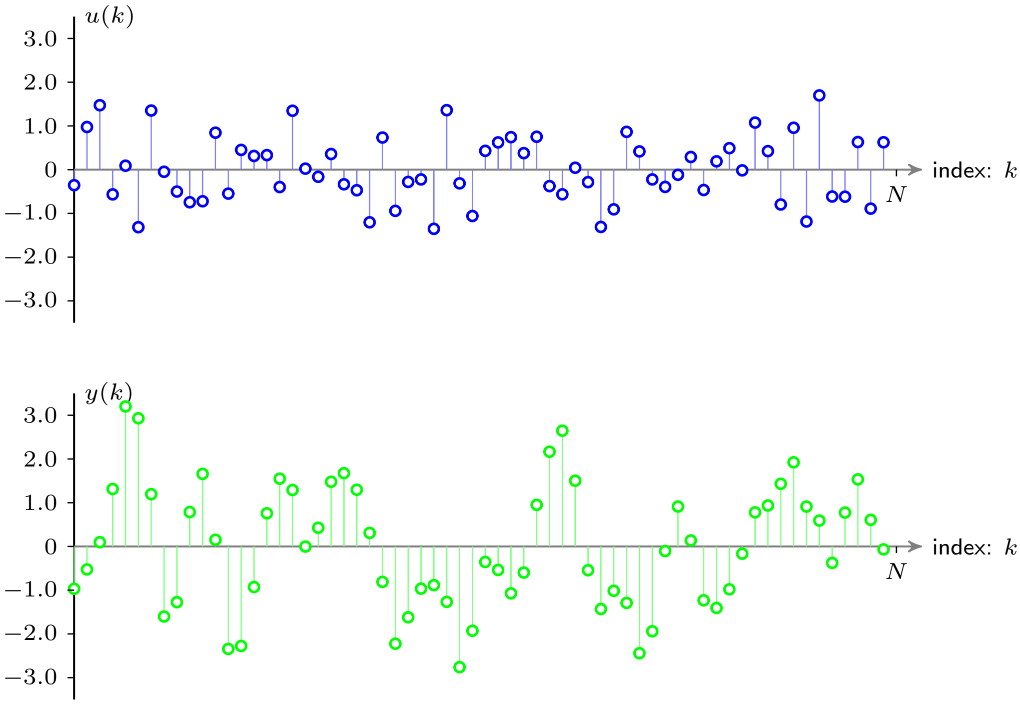
$$R_{yu}(\tau) = \frac{1}{N}\sum_{k=0}^{N-1} y(k)u(k-\tau).$$

Cross-spectral density (FT of the cross-correlation):

$$\phi_{yu}(\mathrm{e}^{j\omega_n}) = \sum_{\tau=0}^{N-1} R_{yu}(\tau)\mathrm{e}^{-j\omega_n\tau}, \quad \omega_n = \tfrac{2\pi n}{N}, \quad n = 0, \ldots, N-1$$

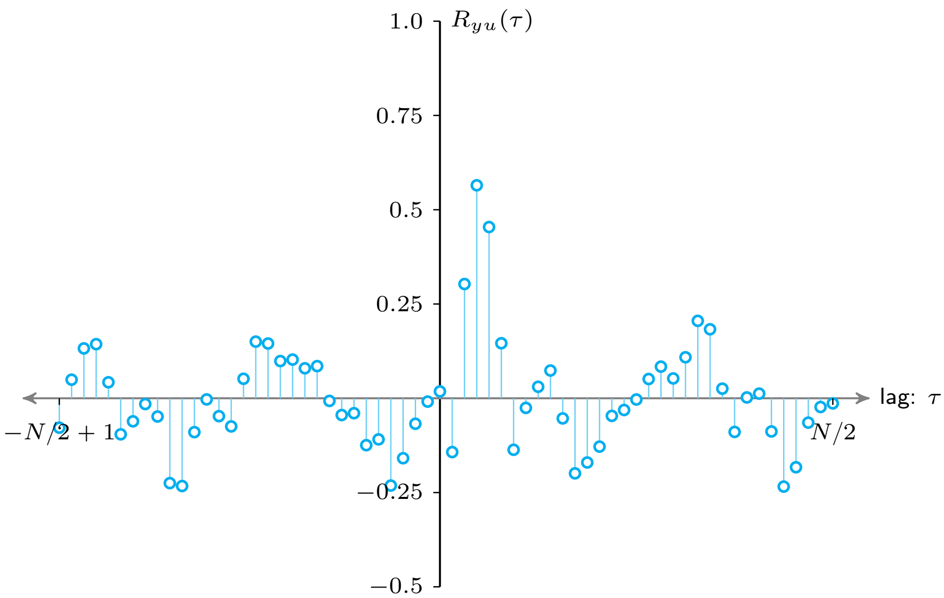
$$= \frac{1}{N} Y(\mathrm{e}^{j\omega_n})U^*(\mathrm{e}^{j\omega_n})$$

Cross-correlation example (periodic signal, $N = M$)



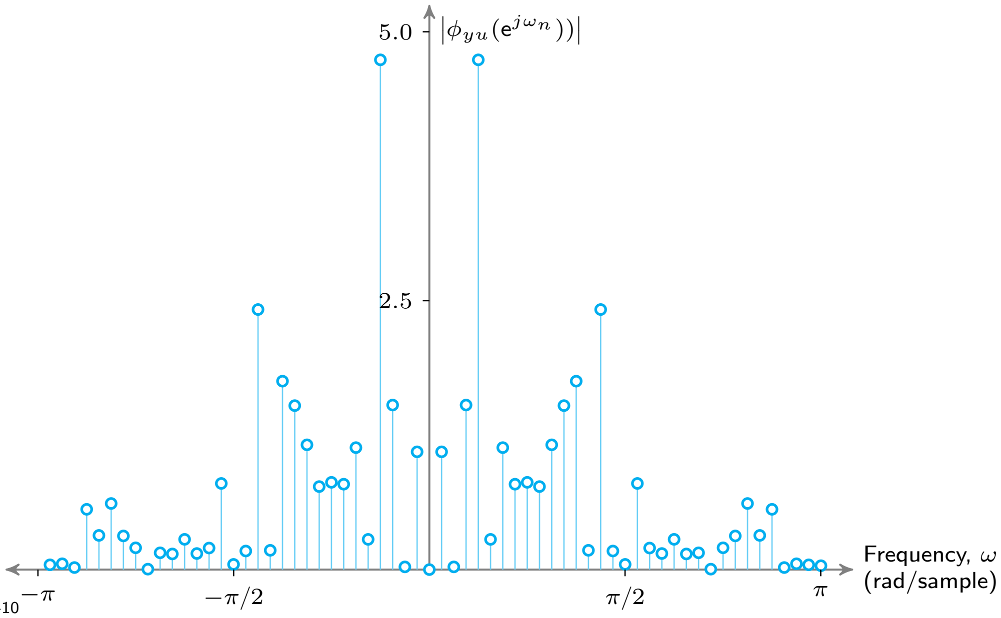
Cross-correlation example (periodic signal, $N = M$)

$$R_{yu}(\tau) = \frac{1}{N} \sum_{k=0}^{N-1} y(k)u(k-\tau), \quad \tau = -N/2+1, \dots, N/2.$$



Cross power spectral density example (periodic signal, $N = M$)

$$\phi_{yu}(\mathrm{e}^{j\omega_n}) = \sum_{\tau=0}^{N-1} R_{yu}(\tau)\mathrm{e}^{-j\omega_n \tau}, \quad \omega_n = \frac{2\pi n}{N}, \quad n = -N/2+1, \ldots, N/2.$$

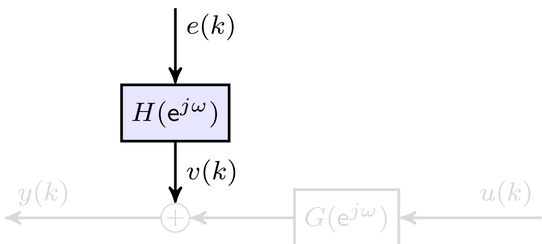


Noise models: random signals

Normally distributed noise:

$$e(k) \in \mathcal{N}(0, \sigma^2), \implies \begin{cases} E\{e(k)\} = 0 & \text{(zero mean)} \\ E\{|e(k)|^2\} = \sigma^2 & \text{(variance)} \end{cases}$$

The $e(k)$ are independent and identically distributed (i.i.d.).



$$v(k) = \sum_{l=0}^{\infty} h(l)e(k-l) = H\, e(k) \quad \text{with } e(k) \in \mathcal{N}(0, \sigma^2).$$

Autocorrelation and autocovariance (random signals)

Correlation

Define the autocorrelation function as;

$$R_x(\tau) = E\{x(k)x(k-\tau)\}$$
$$= E\{x(k)x^*(k-\tau)\} \qquad \text{(in the complex multivariable case)}$$

($x(k)$ assumed to be stationary)

Covariance

The covariance function is defined as:

$$R_x(\tau) = E\{(x(k) - E\{x\})(x(k-\tau) - E\{x\})\}$$

We will assume that random signals are zero mean and use the notation $R_x(\tau)$ for both the autocorrelation and covariance functions.

Power spectral density (random signals)

The *power* spectral density is defined as the Fourier transform of the autocovariance, $R_x(\tau)$,

$$\phi_x(e^{j\omega}) := \sum_{\tau=-\infty}^{\infty} R_x(\tau)e^{-j\omega\tau} \quad \text{where} \quad \omega \in [-\pi, \pi).$$

The inverse transform is given by,

$$R_x(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_x(e^{j\omega})e^{j\omega\tau}d\omega.$$

For a zero-mean random signal,

$$\lim_{N\longrightarrow\infty} \frac{1}{N}\sum_{k=0}^{N-1} |x(k)|^2 = \text{var}(x(k)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_x(e^{j\omega})\,d\omega$$

Basic properties

Autocovariance:

$$R_x(-\tau) = R_x^*(\tau)$$
$$R_x(0) \geqslant |R_x(\tau)| \quad \text{for all } \tau > 0$$

Spectral density:

$$\phi_x(\mathrm{e}^{j\omega}) \in \mathcal{R}$$
$$\phi_x(\mathrm{e}^{j\omega}) \geqslant 0 \quad \text{for all } \omega$$
$$\phi_x(\mathrm{e}^{j\omega}) = \phi_x(\mathrm{e}^{-j\omega}) \quad \text{for all real-valued } x(k)$$

Cross-covariance (random signals)

For random $y(k)$ and $u(k)$, the cross-covariance is:

$$R_{yu}(\tau) = E\left\{(y(k) - E\{y(k)\})(u(k-\tau) - E\{u(k)\})\right\}$$

For zero mean signals, $E\{y(k)\} = 0$ and $E\{u(k)\} = 0$, this is equal to the cross-correlation,

$$R_{yu}(\tau) = E\{y(k)u(k-\tau)\}$$

Joint stationarity is required to make the definition dependent on τ alone.

If $R_{yu}(\tau) = 0$ for all τ then $y(k)$ and $u(k)$ are *uncorrelated*.

Cross power spectral density (random signals)

The Fourier transform of the cross-covariance, $R_{yu}(\tau)$, is defined as the cross spectral density, or cross-spectrum,

$$\phi_{yu}(\mathrm{e}^{j\omega}) = \sum_{\tau=-\infty}^{\infty} R_{yu}(\tau)\mathrm{e}^{-j\omega\tau}, \quad \omega \in [-\pi, \pi).$$

The inverse is,

$$R_{yu}(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_{yu}(\mathrm{e}^{j\omega})\mathrm{e}^{j\omega\tau} d\omega.$$

Discrete-Fourier Transform (finite-length signals)

Finite length signal,

$$x(k), \quad k = 0, \ldots, K-1.$$

We take our calculation length to be the entire signal ($N = K$),

The Discrete Fourier Transform (DFT) of $x(k)$ is:

$$X_N(\mathrm{e}^{j\omega_n}) = \sum_{k=0}^{N-1} x(k)\mathrm{e}^{-j\omega_n k}, \quad \text{where} \quad \omega_n = \frac{2\pi n}{N}, \quad n = 0, \ldots, N-1.$$

The inverse DFT is,

$$x(k) = \frac{1}{N}\sum_{n=0}^{N-1} X_N(\mathrm{e}^{j\omega_n})\mathrm{e}^{j\omega_n k}, \quad k = 0, \ldots, N-1.$$

Periodogram

The periodogram (for a random signal $v(k)$) is defined as:

$$\frac{1}{N}\left|V_N(\mathrm{e}^{j\omega})\right|^2$$

See [Schuster, 1900] for an interesting application.

Asymptotically unbiased estimator of the spectrum:

$$\lim_{N\longrightarrow\infty} E\left\{\frac{1}{N}|V_N(\mathrm{e}^{j\omega})|^2\right\} \;=\; \phi_v(\omega)$$

This assumes that the autocorrelation decays quickly enough:

$$\lim_{N\longrightarrow\infty}\frac{1}{N}\sum_{\tau=-N}^{N}|\tau R_v(\tau)| = 0$$

Bibliography

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