Notational nightmares; the wide variety of notations and definitions in System Identification and Spectral Analysis

Roy Smith

September 9, 2020

1 Why?

There are a number of different formulations for the most common concepts in spectral analysis and system identification. Some of these may have come from the differing background of the early participants; others may arise from the need to improve or standardise the approaches.

Without wishing to pass judgment on the authors' motivations, I will simply attempt to provide a summary of the various formulations relevant to a system identification course. Although the reader may wish to select a favourite, reading the literature forces her or him to at least recognise the others.

The focus here is on the estimating the properties of discrete-time signals. These may come from underlying continuous-time signals or may simply exist solely in the discrete domain.

2 The Sources

This is not an exhaustive listing, nor is it intended as a recommended reading list. These are the texts on my bookshelf. For simplicity the texts and papers are abbreviated as:

2.1 Texts

O&W Alan V. Oppenhiem & Alan S. Wilsky with S.Hamid Nawab, *Signals & Systems,* Prentice-Hall, 2nd Ed., 1996.

O&S Alan V. Oppenhiem & Ronald W. Schafer, *Digital Signal Processing,* Prentice-Hall, 1975.

LL Lennart Ljung, *System Identification; Theory for the User,* Prentice-Hall, 2nd Ed., 1999.

S&M Petre Stoica & Randolph Moses, *Introduction to Spectral Analysis,* Simon & Schuster, 1997.

2.2 Papers

PW Peter Welch, "The use of the fast Fourier transform for the estimation of power spectra: A method based on time averaging over short, modified periodograms," *IEEE Trans. Audio and Electroacoustics,* vol. 15(2), pp. 70–73, 1967.

3 Fourier and Transform Concepts

3.1 Local notation

For at least this document we denote discrete-time signals, by

$$x(k), \quad \text{where} \quad \begin{cases} k = 0, \ldots, K-1 & \text{for finite length signals, or} \\ k = 0, \ldots, \infty & \text{for infinite signals, or} \\ k = -\infty, \ldots, 0, \ldots, \infty & \text{for doubly infinite signals.} \end{cases}$$

When considering a finite length signal, we will use K to denote its length (the number of sampled time points in the signal). When making a calculation (for example: a discrete Fourier transform) we must choose a "calculation length" which we will denote by N. In many instances we choose to do the calculation over all of our available data (i.e. $N = K$), but keep in mind that there are cases when it is better to choose $N < K$.[1]

In the following $X(n)$ denotes the frequency domain representation, written here as a function of its discrete index, n. As the index is associated with a particular frequency it can also be written as

$$X(\mathrm{e}^{j\omega_n}), \quad \text{with} \quad \omega_n = \frac{2\pi n}{N}.$$

Although it is simpler to write this as $X(\omega)$, the exponent form will be kept to emphasise the periodic nature of the functions we are interested in.

3.2 Z-transform

$$X(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k}$$

The only significant variation in the Z-transform is in the way it is pronounced.

3.3 Discrete Time Fourier Transform

The discrete-time Fourier Transform, and its inverse are defined by,

$$X(\mathrm{e}^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k) \mathrm{e}^{-j\omega k}$$

$$x(k) = \frac{1}{2\pi} \int_{2\pi} X(\mathrm{e}^{jw}) \mathrm{e}^{j\omega k} d\omega$$

Note that $X(\mathrm{e}^{j\omega})$ is a periodic function of a continuous variable, ω. Usually only one period is presented. It could be between 0 and 2π or between $-\pi$ and π, or simply the non-negative frequencies, 0 to π. Linear frequency scales are common in signal processing, logarithm frequency scales are more common in control systems.

This definition is used by **O&S**, **O&W**, **S&M**.

[1] One example is when we have a periodic excitation. It is almost always better to choose N to be an integer number of periods, even if that involves throwing away some of our data.

3.4 Discrete Fourier Transform

This applies to a finite length signals and here we take the calculation length to be N. The usual approach is to define the Discrete Fourier Transform (DFT) as the Fourier Series of the signal's period continuation. Variations in scale factors arise at this point.

In **O&W**[2] and **PW** we find,

$$X(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(k)\mathrm{e}^{-j(2\pi/N)kn}, \quad n = 0, \ldots, N-1.$$

$$x(k) = \sum_{n=0}^{N-1} X(n)\mathrm{e}^{j(2\pi/N)kn}, \quad k = 0, \ldots, N-1.$$

This leads to a scale factor of $1/N$ between the DFT and the Fourier Series.

In **LL** we find,

$$X(n) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} x(k)\mathrm{e}^{-j(2\pi/N)kn}, \quad n = 0, \ldots, N-1.$$

$$x(k) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} X(n)\mathrm{e}^{j(2\pi/N)kn}, \quad k = 0, \ldots, N-1.$$

In **O&S**[3] and **S&M** we find,

$$X(n) = \sum_{k=0}^{N-1} x(k)\mathrm{e}^{-j(2\pi/N)kn}, \quad n = 0, \ldots, N-1.$$

$$x(k) = \frac{1}{N}\sum_{n=0}^{N-1} X(n)\mathrm{e}^{j(2\pi/N)kn}, \quad k = 0, \ldots, N-1.$$

In this case the DFT matches the Fourier Series of the periodic continuation of the signal.

This last convention matches the `fft` and `ifft` commands in MATLAB. Note however that as MATLAB doesn't support zero as an indexing variable the calculation indices are shifted by one.

[2] To be fair this only appears in an exercise.

[3] It is interesting that one author has used different definitions for different textbooks.

All of the major concepts involving the DFT work with any of these scalings. However we must be careful when proving theorems and deriving related results such as Parseval's theorem.

4 Spectral analysis concepts

4.1 Autocorrelation and autocovariance

In the statistics literature the *autocorrelation* is frequently defined (for a stationary signal) as,

$$R(\tau) = \frac{E\{(x(k) - \mu_x)(x(k+\tau) - \mu_x)\}}{\sigma_x^2},$$

where the mean and variance of the distribution from which $x(k)$ is drawn are μ_x and σ_x^2 respectively.

In signal processing literature the definition of the *autocorrelation* does not subtract the mean or scale by the inverse of the variance. In our text examples we have:

In both **O&S**[4] and **O&W** we have,

$$R(\tau) = \sum_{k=-\infty}^{\infty} x(k)x(k+\tau).$$

This definition is only applicable to finite energy signals. In particular, it is not applicable to random signals or periodic signals.

In **S&M** we have,

$$R(\tau) = \sum_{k=-\infty}^{\infty} x(k)x(k-\tau).$$

Note that there is a difference in the direction of the signal shift. For the autocorrelation this makes no difference—it does affect the cross-correlation definitions though.

For random signals most authors use the term *autocovariance* and define it (for stationary signals) as follows:

[4] In **O&S** this called an *aperiodic autocorrelation.*

In **LL** we have (for stationary zero mean random signals),

$$R(\tau) = E\{(x(k)(x(k-\tau)\}.$$

In fact **LL** does not define or use a correlation function in this context at all; it only appears as the title of the correlation method for analysing single sinusoidal excitation.

Again **O&S** changes the sign of τ and also subtracts off the mean, to give the *autocovariance* definition as,

$$R(\tau) = E\{(x(k) - \mu_x)(x(k+\tau) - \mu_x)\}.$$

O&S make the observation that for zero mean signals the *autocorrelation* and *autocovariance* are the same. Note that when defining *covariance* or *autocovariance* functions there is no scaling by the inverse of the variance.

4.2 Periodograms

The periodogram, denoted here by $I(\mathrm{e}^{j\omega})$, can also defined as a periodic function of a continuous frequency variable.

In **O&S** and **S&M**[5] we find,

$$I(\mathrm{e}^{j\omega}) = \frac{1}{N}\left|\sum_{k=0}^{N-1} x(k)\mathrm{e}^{-j\omega k}\right|^2.$$

In **LL**[6] we find,

$$I(\mathrm{e}^{j\omega}) = \left|\sum_{k=0}^{N-1} x(k)\mathrm{e}^{-j\omega k}\right|^2.$$

In **PW** the periodogram is given for averages but the base definition can be deduced as,

$$I(\mathrm{e}^{j\omega}) = N\left|\sum_{k=0}^{N-1} x(k)\mathrm{e}^{-j\omega k}\right|^2.$$

[5] In **S&M** the periodogram shares the same symbol as the spectrum.
[6] In **LL** the periodogram is not given its own symbol.

These texts are internally consistent as the scalings on the periodogram are required to obtain convergence of the periodogram to the spectrum without introducing a further scaling term.

4.3 Window functions

Windowing is used to smooth frequency domain estimates of spectra or frequency response functions. Windows can be applied in the time-domain to autocorrelation estimates or to signals before performing DFT calculations. Window functions therefore generally have a frequency domain and time-domain representation.

We will write time-domain windows as a function of a width parameter, γ, and the lag variable, τ, in the form, $w_\gamma(\tau)$. Many (but not all) time-domain window definitions can be written in terms of the ratio of these parameters,

$$w_\gamma(\tau) = f(\tau/\gamma).$$

There are several conflicting ways in which these definitions are used. The use of window functions in the time-domain appears to be more consistent than in the frequency domain. Again the variation is primarily in the scaling, with a factor of 2π appearing differently in different texts.

In **LL** the time-domain window function is used in calculating a smoothed spectral estimate is via,

$$\phi(\mathrm{e}^{j\omega}) = \sum_{\tau=-\infty}^{\infty} w_\gamma(\tau) R(\tau) \mathrm{e}^{-j\tau\omega},$$

where $R(\tau)$ is the autocorrelation (or an estimate of the autocorrelation). Note that this equation simultaneously performs the smoothing and Fourier transform to return a result in the frequency domain.

In the frequency domain the window is used to smooth the data via,

$$\phi(\mathrm{e}^{j\omega}) = \int_{-\pi}^{\pi} W_\gamma(\mathrm{e}^{j\xi} - \mathrm{e}^{j\omega}) \hat{\phi}(\mathrm{e}^{j\omega}) d\xi,$$

where $\hat{\phi}(\mathrm{e}^{j\omega})$ is the unsmoothed frequency domain function. The shift in $W_\gamma(\omega)$ is due to the fact that $W_\gamma(\mathrm{e}^{j\omega})$ is centred around $\omega = 0$.

The two types of smoothing are equivalent (at least as $N \longrightarrow \infty$) if we define the relationship between the windows as,

$$w_\gamma(\tau) = \int_{-\pi}^{\pi} W_\gamma(\mathrm{e}^{j\xi})\mathrm{e}^{j\xi\tau} d\xi.$$

This differs from the usual inverse Fourier Transform by a factor of $1/2\pi$.

In **S&M** (and briefly in **O&S**) the time domain windowed spectral estimate is defined identically,

$$\phi(\mathrm{e}^{j\omega}) = \sum_{\tau=-\infty}^{\infty} w_\gamma(\tau) R(\tau) \mathrm{e}^{-j\tau\omega},$$

However, the frequency domain weighting is given by,

$$\phi(\mathrm{e}^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W_\gamma(\mathrm{e}^{j\omega} - \mathrm{e}^{j\xi}) \hat{\phi}(\mathrm{e}^{j\omega}) d\xi,$$

making the relationship between the time- and frequency-domain windows the usual Fourier Transform,

$$w_\gamma(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W_\gamma(\mathrm{e}^{j\xi})\mathrm{e}^{j\xi\tau} d\xi.$$

Note also that the frequency domain smoothing is presented as a convolution in **S&M** and a shift in **LL**. As frequency domain windows are almost always symmetric this doesn't make a mathematical difference.

These notational discrepancies mean that for the same time-domain window, the frequency domain versions in **S&M** are 2π times larger than those in **LL**.

The texts themselves are internally consistent, but applying a window definition from one text to a periodogram definition from another may lead to a scaling error. Care is required in translating these concepts.