Notes on SISO controllability limitations

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This document is meant to be an informal note summarising some controllability limitations in SISO systems that arise because of the plant open loop characteristics, i.e. RHP poles or zeros, delays, or because of particular disturbance rejection or input saturation requirements.

RHP Zeros and Poles

The limitations in the presence of RHP zeros and poles arise from the maximum modulus principle

Theorem 1 (Maximum modulus priciple)**.** *Given a function $H : \mathbb{C} \to \mathbb{C}$ analytic on the RHP, then*

$$\sup_{s \in RHP} |H(s)| = \sup_{\omega \in \mathbb{R}} |H(j\omega)|$$

□

In other words, if a complex function has no poles in the RHP, then the maximum modulus on the RHP is attained on the $j\omega$ axis. We use this to derive the limitations for RHP zeros and poles.

RHP Zeros [Skogestad sec. 5.7]

IF z is a RHP zero of G, then $S(z) := \frac{1}{1+K(z)G(z)} = 1$. This is because we cannot cancel the RHP zero of G with a pole in K. Furthermore, if K is stabilising, then S is analytic in the RHP (i.e. stable). To obtain good performance we want

$$\|w_P S\|_\infty \leq 1. \tag{1}$$

for some w_P that captures our performance specifications. Then we have:

$$1 \geq |w_P(j\omega)S(j\omega)| \geq |w_P(z)S(z)| \geq |w_P(z)|,$$

where the second inequality follows from the maximum modulus principle and the third from $S(z) = 1$. We therefore get that in order to satisfy (1), we **must** have that $\underline{|w_P(z)| \leq 1}$.

If we take a typical choice for w_P as:

$$w_P := \frac{s/M + \omega_B}{s + \omega_B A}$$

with $A = 0$ and $M = 2$ (zero steady state error and maximum peak = 2), then we get the familiar condition on the cutoff frequency.

$$\left| \frac{z/2 + \omega_B}{z} \right| \leq 1 \iff \omega_B \leq \frac{z}{2}.$$

RHP Poles [Skogestad sec. 5.9]

If p is a RHP pole of G, then $T(p) := \frac{K(p)G(p)}{1+K(p)G(p)} = 1$ (in a limit sense). This is because we cannot cancel the RHP pole of G with a zero in K. Furthermore, if K is stabilising, then T is analytic in the RHP (i.e. stable). To obtain noise rejection and robust stability we normally specify that

$$\|w_I T\|_\infty \leq 1. \tag{2}$$

for some w_I that encompasses our design needs. Then, similarly as before, we have:

$$1 \geq |w_i(j\omega)T(j\omega)| \geq |w_I(p)T(p)| \geq |w_I(p)|,$$

where the second inequality follows from the maximum modulus principle and the third from $T(p) = 1$. We therefore get that in order to satisfy (2), we **must** have that $\underline{|w_I(p)| \leq 1}$.

If we take a typical choice for w_I as:

$$w_I := \frac{s}{\omega_{BT}} + \frac{1}{M_T},$$

with $M_T = 2$, then we get the familiar condition on the cutoff frequency.

$$w_I := \frac{p}{\omega_{BT}} + \frac{1}{2} \leq 1 \iff \omega_{BT} \geq 2p.$$

Disturbance Rejection and Reference Tracking with no Input Saturations [Skogestad sec. 5.10]

The following limitations concern reference tracking and disturbance rejection requirements. They appear as hard constraints on S and thus limit the possible design choices.

Disturbance Rejection

We want to have $|e(\omega)| \leq 1$ for all disturbance $|d(\omega)| \leq 1$ (no requirements on input saturation). For a typical feedback loop with additive disturbance and disturbance transfer function G_d we have that:

$$e = \frac{1}{1+L} G_d d = S G_d d.$$

Then we have disturbance rejection <u>if and only if</u> $\|SG_d\|_\infty \leq 1$.

Reference Tracking

We want to have $|e(\omega)| \leq 1$ for all references $|r(\omega)| \leq 1$ (no requirements on input saturation). For a typical feedback loop with additive disturbance and reference transfer function R we have that:

$$e = \frac{1}{1+L} Rr = SRr.$$

Then we have reference tracking rejection <u>if and only if</u> $\|SR\|_\infty \leq 1$.

Disturbance Rejection and Reference Tracking + Input Saturation [Skogestad sec. 5.11]

The following limitations concern the possibility of satisfying a disturbance rejection/reference tracking requirement while maintaining the input "small". We derive **necessary** conditions that, if violated, guarantee that there exist not controller that achieves such goals.

Disturbance Rejection + Input Saturation

We want to have $|e(\omega)| \leq 1$ for all disturbance $|d(\omega)| \leq 1$, while keeping $|u(\omega)| \leq 1$. To derive the limitations we assume **perfect control**, i.e. we assume that we have perfect knowledge of d and we can directly manipulate the input. In that case $e = y = G_d d + Gu$. If $|G_d(j\omega)| \leq 1$ by setting $u = 0$ we satisfy the specification. If $|G_d(j\omega)| > 1$, since Since we cannot control d, the best we can do is to chose u such that there is a 180° in the complex plane between $G_d d$ and Gu so that we keep $|e|$ as small as possible. In that case we get

$$|e| = |G_d d| - |Gu| \leq 1,$$

since $|d| \leq 1$ we get that

$$|e| \leq |G_d| - |Gu| \leq 1,$$

which implies that

$$|u| \geq |G|^{-1}(|G_d| - 1).$$

Which means that, unless $\underline{|G| \geq |G_d| - 1 \text{ for all } \omega \text{ such that } |G_d(j\omega)| > 1}$, in order to satisfy the disturbance requirement, we need $|u| > 1$ even with perfect control. **Note: This does not mean that if $|G| \geq |G_d| - 1$ for all ω such that $|G_d(j\omega)| > 1$ there exist a controller that achieves the requirement**. This test can only give a negative result in case it is violated.

Reference Tracking + Input Saturation

We want to have $|e(\omega)| \leq 1$ for all references $|r(\omega)| \leq 1$, while keeping $|u(\omega)| \leq 1$. Again, to derive the limitations we assume **perfect control**, i.e. we assume that we have perfect knowledge of r and we can directly manipulate the input. In that case $e = Rr - y = Rr - Gu$. If $|R(j\omega)| \leq 1$ by setting $u = 0$ we satisfy the specification. If $|R(j\omega)| > 1$, since we cannot control r, the best we can do is to chose u such that there is a 180° in the complex plane between $-Rr$ and Gu so that we keep $|e|$ as small as possible. In that case we get

$$|e| = |Rr| - |Gu| \leq 1,$$

since $|r| \leq 1$ we get that

$$|e| \leq |R| - |Gu| \leq 1,$$

which implies that

$$|u| \geq |G|^{-1}(|R| - 1).$$

Which means that, unless $\underline{|G| \geq |R| - 1 \text{ for all } \omega \text{ such that } |R(j\omega)| > 1}$, in order to satisfy the tracking requirement, we need $|u| > 1$ even with perfect control. **Note: This does not mean that if $|G| \geq |R| - 1$ for all ω such that $|R(j\omega)| > 1$ there exist a controller that achieves the requirement**. This test can only give a negative result in case it is violated.