Control Systems 2

Lecture 13: Digital controller design

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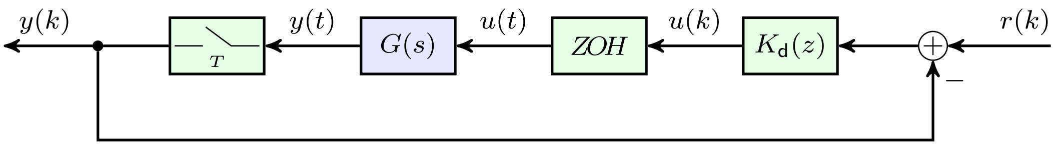
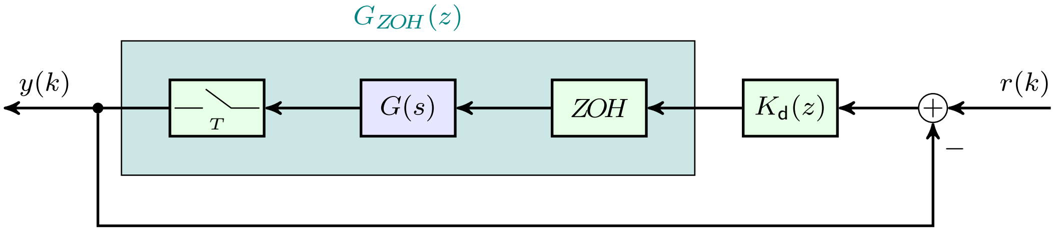
8:15 Wednesday 1st June, 2022

Digital control system design

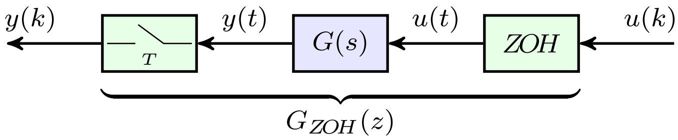
Sampled-data closed-loop



$G_{ZOH}(z)$ equivalence



Zero-order hold equivalence — transfer function



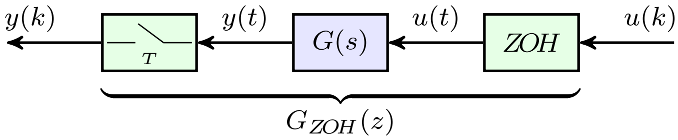
Input: $u(k) = \begin{cases} 1 & k = 0, \\ 0 & k \neq 0 \end{cases}, \quad u(t) = \text{step}(t) - \text{step}(t - T).$

Output: $y(s) = \left(1 - \mathrm{e}^{-Ts}\right) \frac{G(s)}{s}.$

We now sample this, and take the Z-transform,

$$\begin{aligned} G_{ZOH}(z) &= \mathcal{Z}\left\{ \left(1 - \mathrm{e}^{-Ts}\right) \frac{G(s)}{s} \right\} \\ &= (1 - z^{-1})\mathcal{Z}\left\{ \frac{G(s)}{s} \right\}. \end{aligned}$$

Zero-order hold equivalence — state space



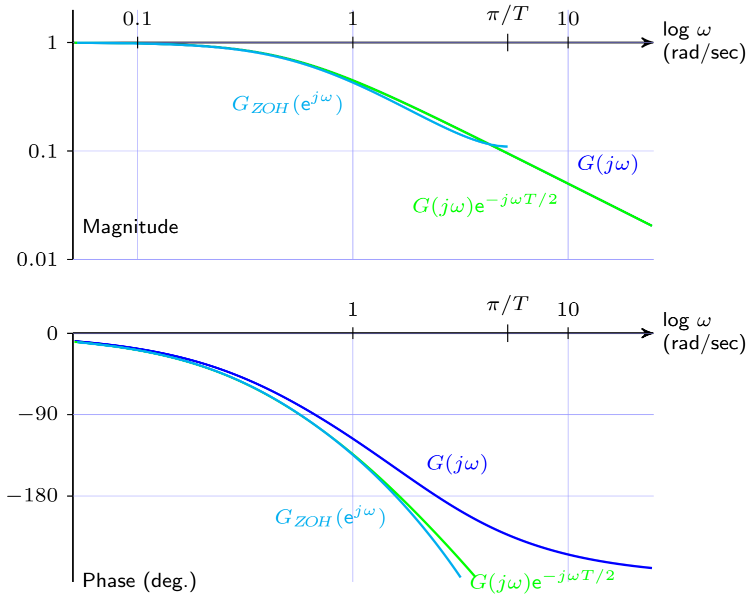
Integrating $\Phi(t)$ over a single sample period (kT to $kT + T$):

$$x(kT + T) = \mathrm{e}^{AT}\, x(kT) + \int_{kT}^{kT+T} \mathrm{e}^{A(kT+T-\tau)}\, Bu(\tau)\, d\tau,$$

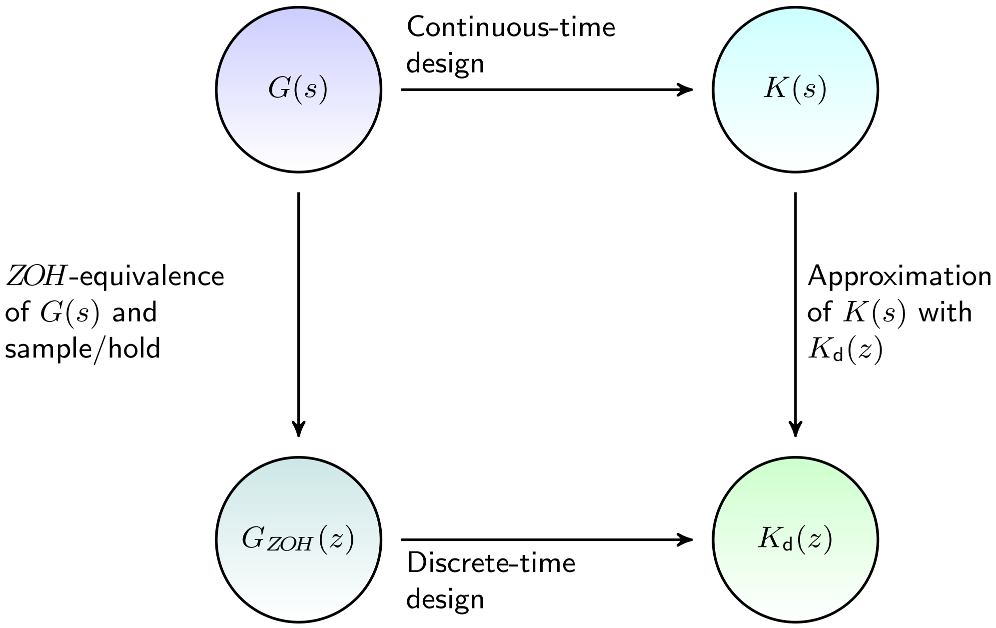
$$\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right] \quad \overset{ZOH\text{-equivalence}}{\Longrightarrow} \quad \left[\begin{array}{c|c} \mathrm{e}^{AT} & \int_0^T \mathrm{e}^{A\eta}\, B\, d\eta \\ \hline C & D \end{array}\right]$$

Zero-order hold equivalence — frequency domain

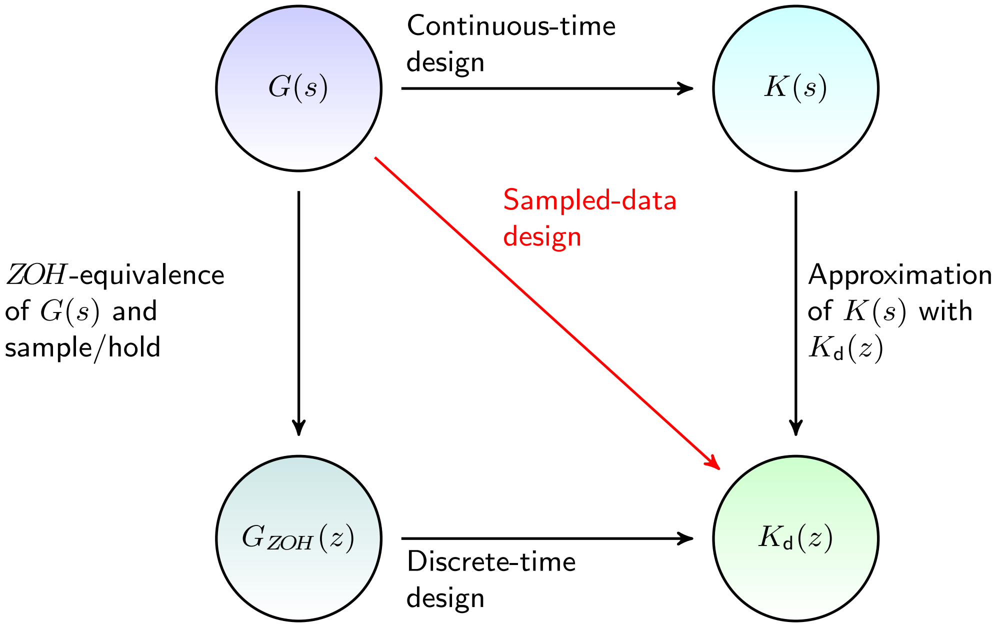
Example: $G(s) = \dfrac{(2-s)}{(2s+1)(s+2)}, \qquad T = 0.6$



Design approaches



Design approaches

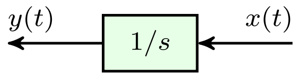


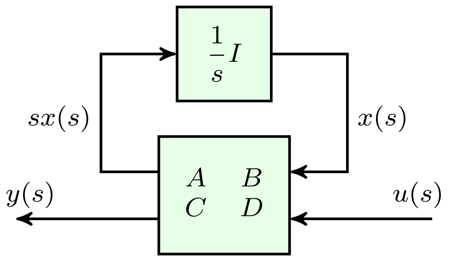
Design by approximation

1. Design a continuous-time controller, $K(s)$
   - Verify stability, performance and bandwidth
   - Verify margins and robustness
2. Select a sample-rate, T
3. Find $K_{\mathsf{d}}(z)$ approximating $K(s)$
4. Calculate the ZOH-equivalent $G_{ZOH}(z)$
5. Check the stability of the $G_{ZOH}(z)$, $K_{\mathsf{d}}(z)$ loop
6. Simulate $K_{\mathsf{d}}(z)$ with $G(s)$ (including sample/hold).
   - Verify simulated performance
   - Examine intersample behaviour

Controller approximation

Approach: approximating the integrators

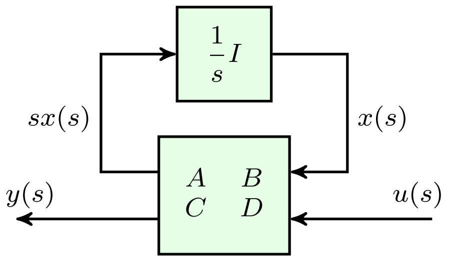




Controller approximation

Approach: approximating the integrators

If $F(z) \approx 1/s$, then, $s \approx F^{-1}(z)$, $\quad \Longrightarrow \; K_{\mathsf{d}}(z) = K(s)\,|_{s=F^{-1}(z)}$



Controller approximation

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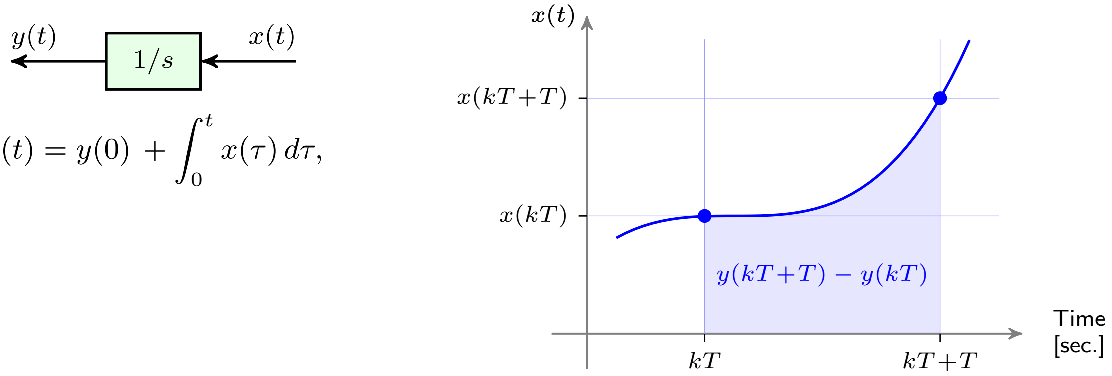
Integration



$$y(t) = y(0) + \int_0^t x(\tau)\, d\tau,$$

The signal, $y(t)$, over a single T second sample period is,

$$y(kT+T) = y(kT) + \int_{kT}^{kT+T} x(\tau)\, d\tau.$$

Trapezoidal approximation

$$\hat{y}(kT+T) = \hat{y}(kT) + Tx(kT) + (x(kT+T) - x(kT))T/2.$$

Taking z-transforms,

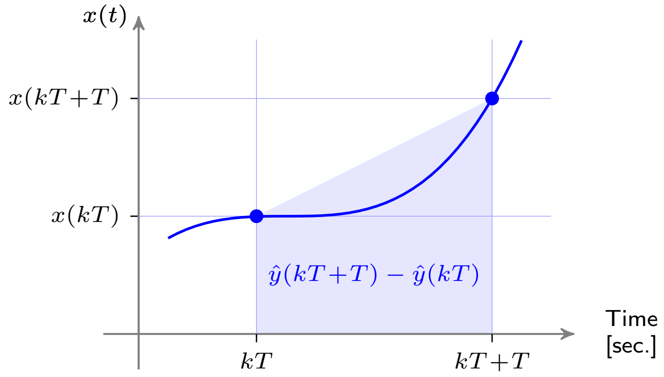
$$z\hat{y}(z) = \hat{y}(z) + Tx(z) + \frac{T}{2}(z-1)x(z),$$

Approximation:

$$\frac{\hat{y}(z)}{x(z)} = F(z) = \frac{T}{2}\frac{z+1}{z-1}.$$

So the substitution is,

$$s \longleftarrow \frac{2}{T}\frac{z-1}{z+1}.$$



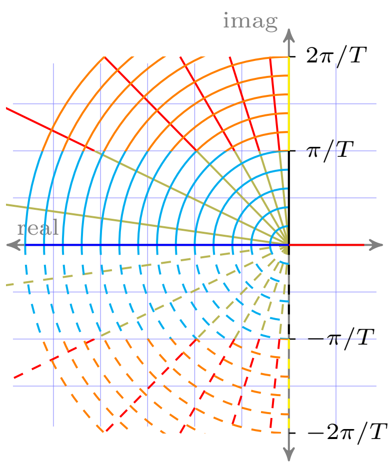
This is known as a bilinear (or Tustin) transform.

Frequency mapping

Pole locations under bilinear transform:

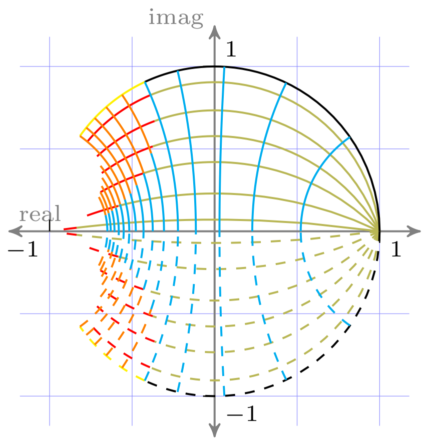
$$\{\, s \mid \mathrm{real}(s) < 0 \,\} \xrightarrow{\text{bilinear}} \{\, z \mid \; |z| < 1 \,\}$$

$K(s)$ stable $\iff$ $K_{\mathsf{d}}(z)$ stable.



$s = \frac{2}{T}\frac{(z-1)}{(z+1)}$

$\longrightarrow$



Bilinear frequency distortion

Ω : discrete-frequencies: $\mathrm{e}^{j\Omega T}$, $\Omega \in (-\pi, \pi]$.

Frequency mapping:

Continuous frequencies, ω to discrete frequencies, Ω.

Substitute $s = j\omega$ and $z = \mathrm{e}^{j\Omega T}$ into $s = \dfrac{2}{T}\dfrac{z-1}{z+1}$:

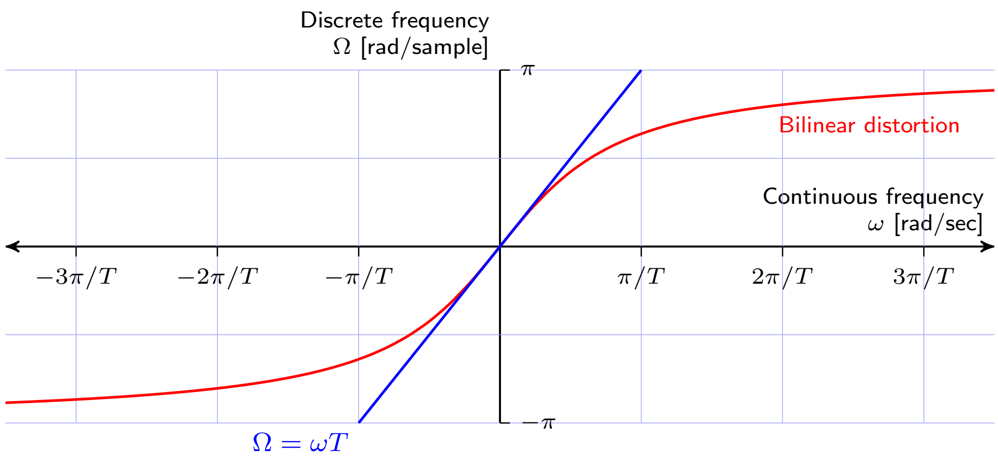
$$j\omega = \frac{2}{T}\frac{\left(1-\mathrm{e}^{-j\Omega T}\right)}{\left(1+\mathrm{e}^{-j\Omega T}\right)} = \frac{2}{T}\frac{j\sin(\Omega T/2)}{\cos(\Omega T/2)} = \frac{2}{T} j\tan(\Omega T/2).$$

Frequency distortion:

$$\Omega = \frac{2}{T}\arctan(\omega T/2)$$

Bilinear frequency distortion

$$\Omega = \frac{2}{T}\arctan(\omega T/2)$$



The $\Omega = \omega T$ line is the sampling mapping.

Prewarping

$$s = \frac{\alpha(z-1)}{(z+1)},\ \alpha \in (0, 2/T),\ \text{maps } \{\text{real}\{s\} < 0\} \text{ to } \{|z| < 1\}.$$

Modifying the frequency distortion

Select a frequency "prewarping frequency", ω_{pw}.

Solve for α such that $K(j\omega_{\text{pw}}) = K_{\text{d}}\left(\mathrm{e}^{j\omega_{\text{pw}}T}\right)$.

The "prewarped" transform makes $K(j\omega) = K_{\text{d}}(\mathrm{e}^{j\omega T})$ at $\omega = 0$ and $\omega = \omega_{\text{pw}}$.

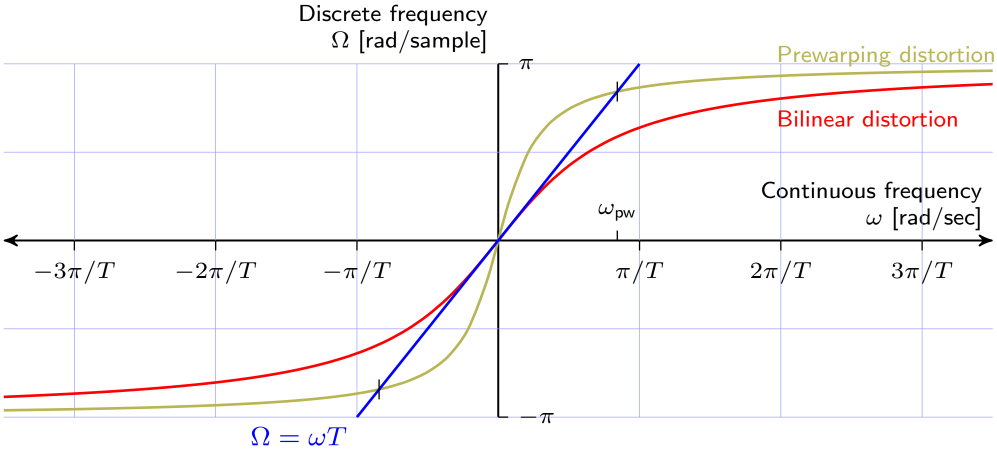
$$j\omega_{\text{pw}} = \frac{\alpha(\mathrm{e}^{j\omega_{\text{pw}}T} - 1)}{(\mathrm{e}^{j\omega_{\text{pw}}T} + 1)} = j\alpha \tan(\omega_{\text{pw}}T/2),$$

which implies that: $\alpha = \dfrac{\omega_{\text{pw}}}{\tan(\omega_{\text{pw}}T/2)}$.

Prewarping

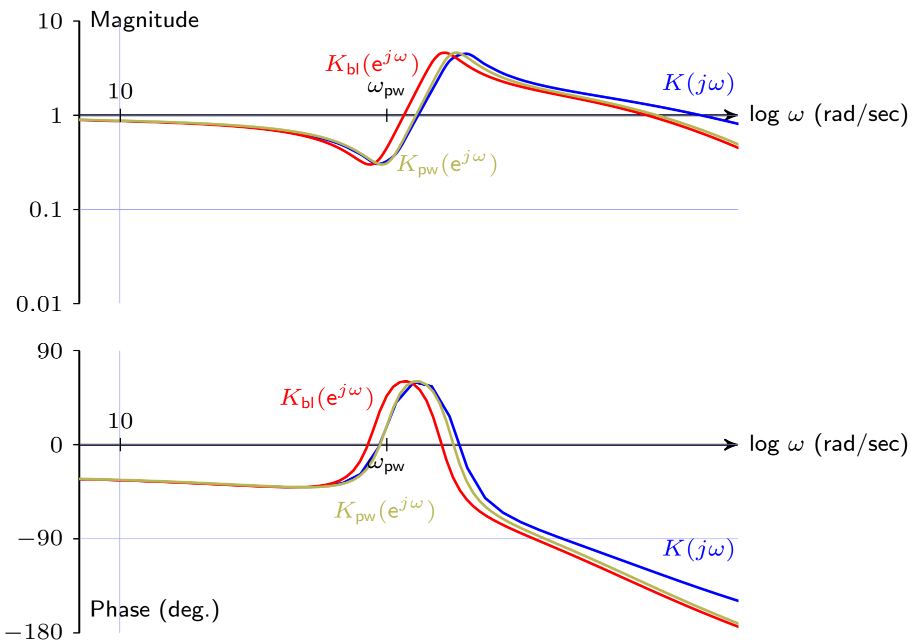
Frequency distortion (bilinear:) $\Omega = \dfrac{2}{T}\arctan(\omega T/2)$.

Frequency distortion (with prewarping): $\Omega = \dfrac{2}{T}\arctan(\omega/\alpha)$



Controller approximations

Bilinear approximation: $K_{\mathsf{bl}}(e^{j\omega})$, Prewarped Tustin approximation: $K_{\mathsf{pw}}(e^{j\omega})$



Choosing a prewarping frequency

The prewarping frequency must be in the range: $0 < \omega_{\mathsf{pw}} < \pi/T$.

- $\alpha = 2/T$ (standard bilinear) corresponds to $\omega_{\mathsf{pw}} = 0$.

Possible options for ω_{pw} depend on the problem:

- The cross-over frequency (helps preserve the phase margin);
- The frequency of a critical notch;
- The frequency of a critical oscillatory mode.

The best choice depends on the most important features in your control design.

Remember: $K(s)$ stable implies $K_{\mathsf{d}}(z)$ stable.

But you **must** check that $(1 + G_{ZOH}(z)K_{\mathsf{d}}(z))^{-1}$ is stable!

Example

Plant model:

$$G(s) = \frac{5(1 - s/z_{\text{rhp}})}{(1+\tau s)} \frac{(s^2 + 2\zeta\eta\omega_m s + \eta^2\omega_m^2)}{(s^2 + 2\zeta\omega_m s + \omega_m^2)} \frac{1}{\eta^2}$$

where $\tau = 0.5$, $z_{\text{rhp}} = 70$, $\omega_m = 20$, $\zeta = 0.05$, and $\eta = 1.2$.

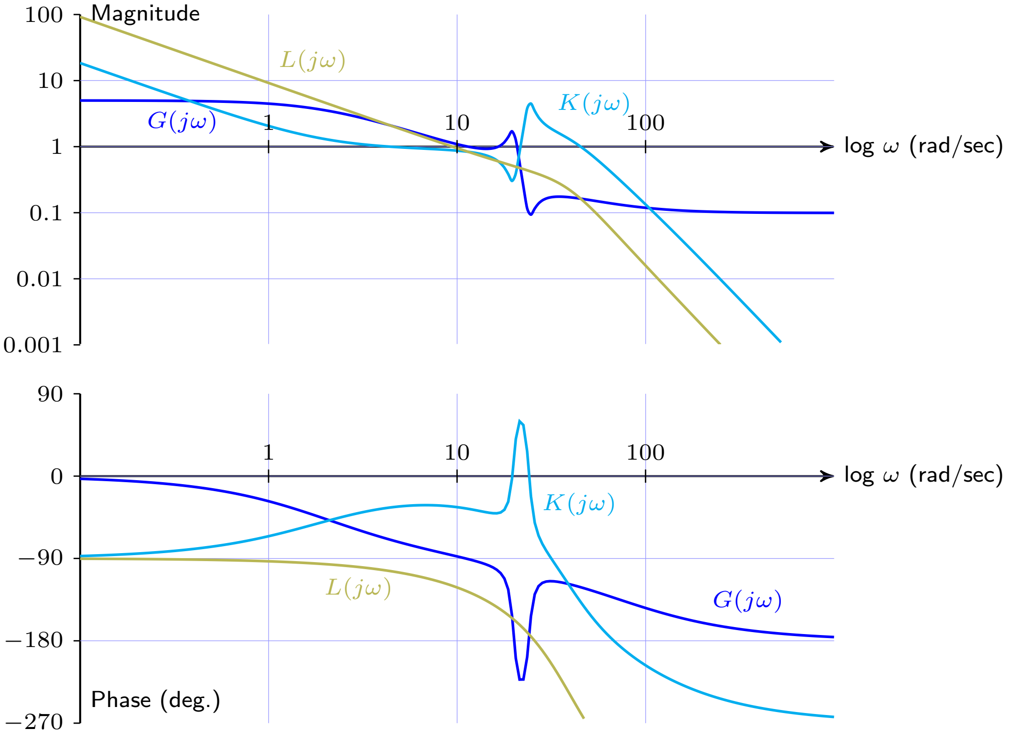
IMC desgn

$T_{\text{ideal}}(s) =$ 3rd order Butterworth filter with bandwidth: 25 [rad./sec.]

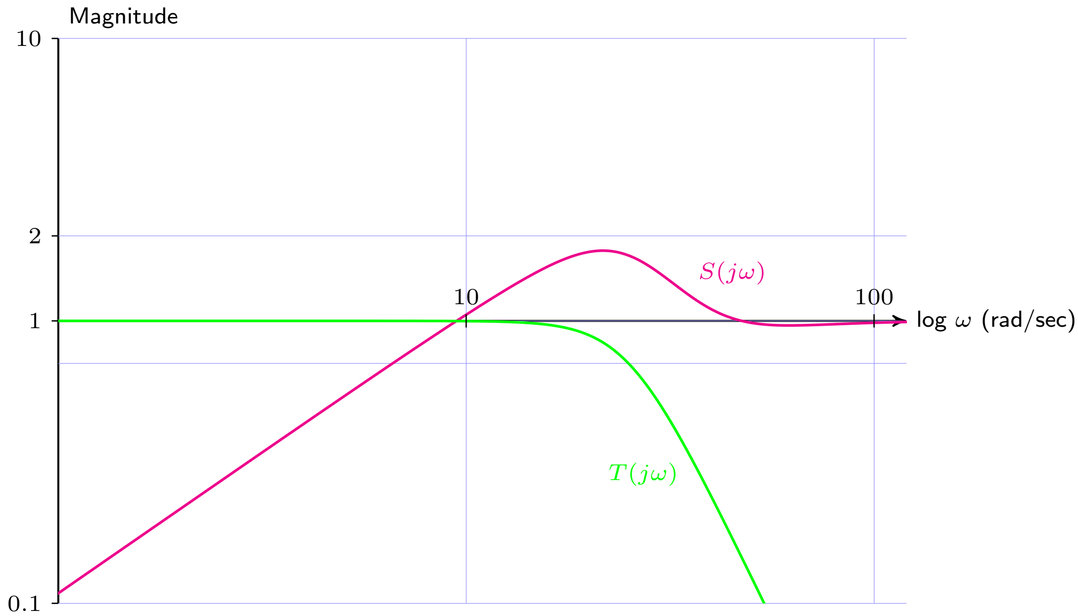
$Q(s) = T_{\text{ideal}}(s) G_{\text{mp}}^{-1}(s)$

$K(s) = (I - Q(s)G(s))^{-1} Q(s).$

Loopshapes

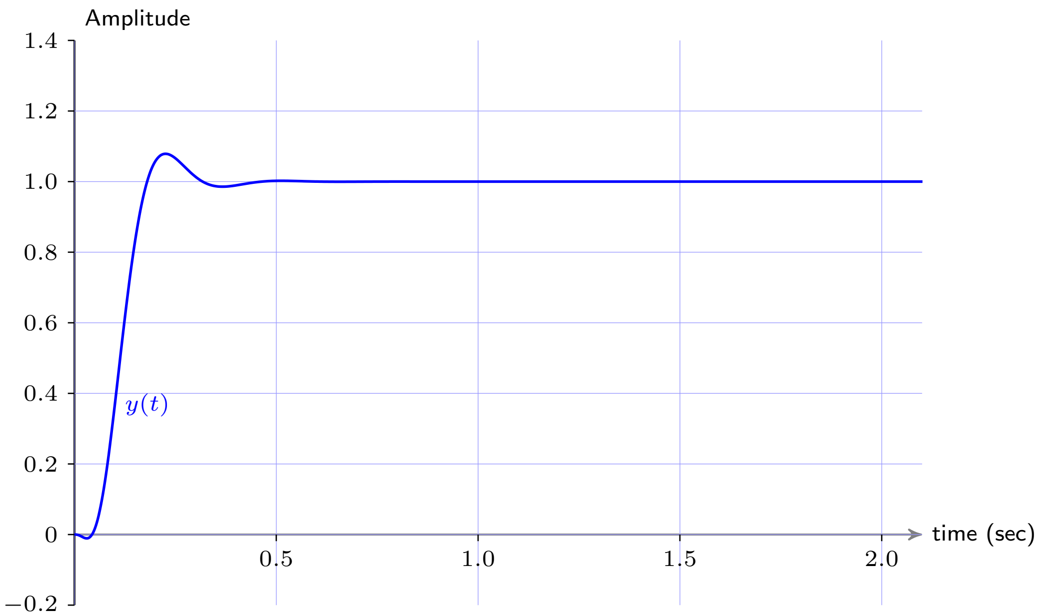


Sensitivity and complementary sensitivity



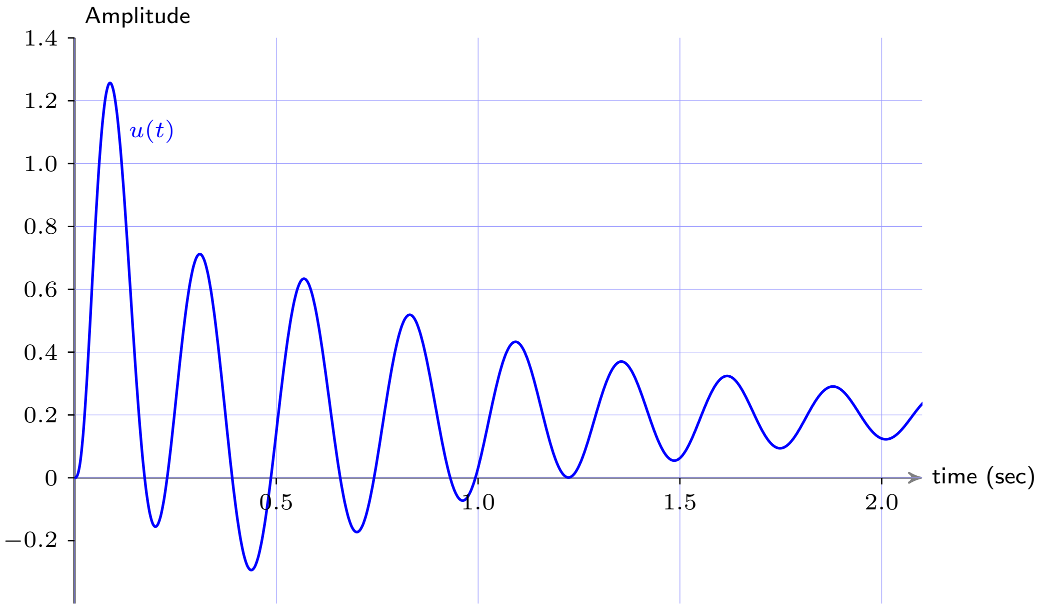
Step response

Output



Step response

Actuation



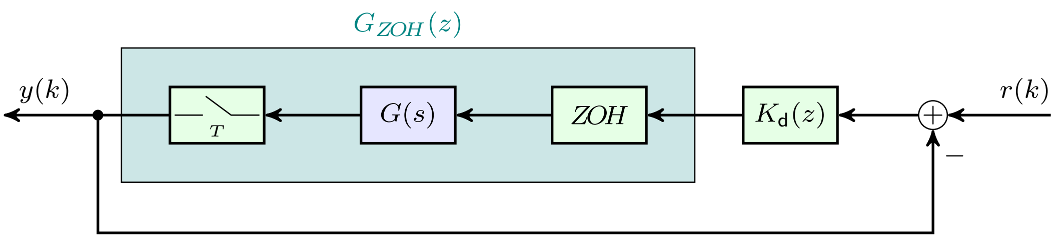
Bilinear/Trapezoidal/Tustin transform

Bilinear transform

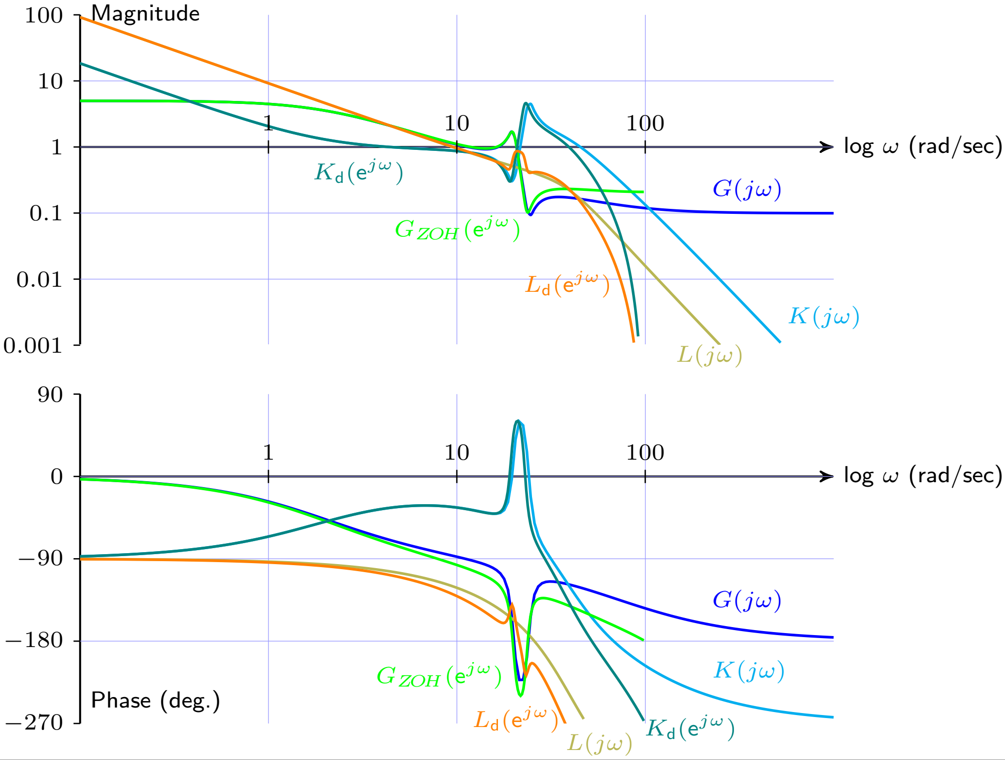
Nyquist frequency: 100 radians/second $\quad\Longrightarrow\quad T = \dfrac{\pi}{100}.$

$$K_{\mathsf{bl}}(z) = K(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}}$$

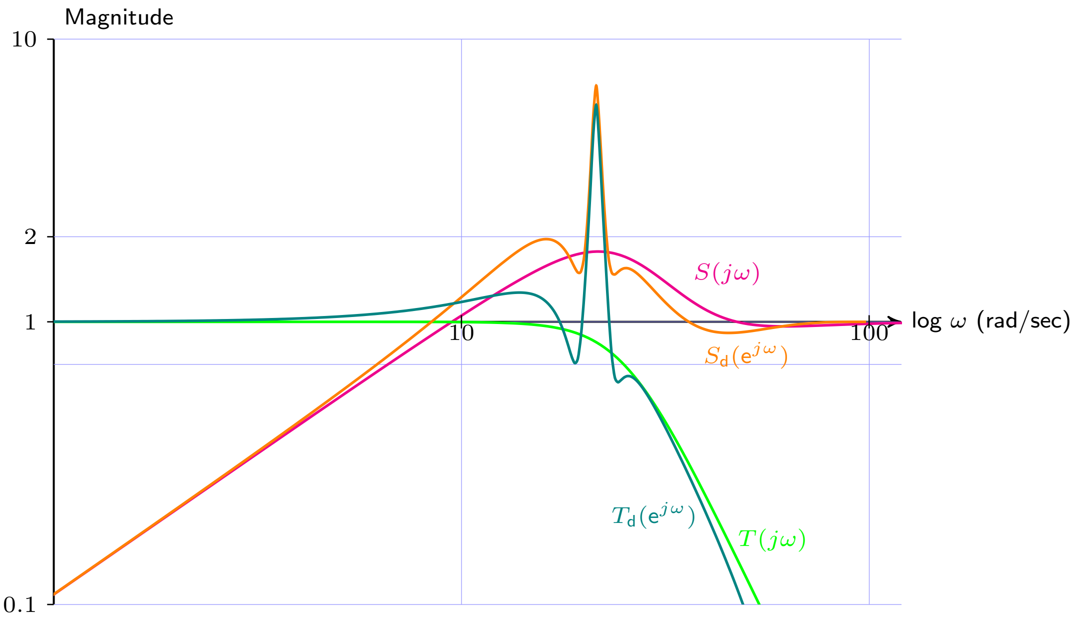
Discrete-time analysis



Loopshapes: bilinearly transformed controller

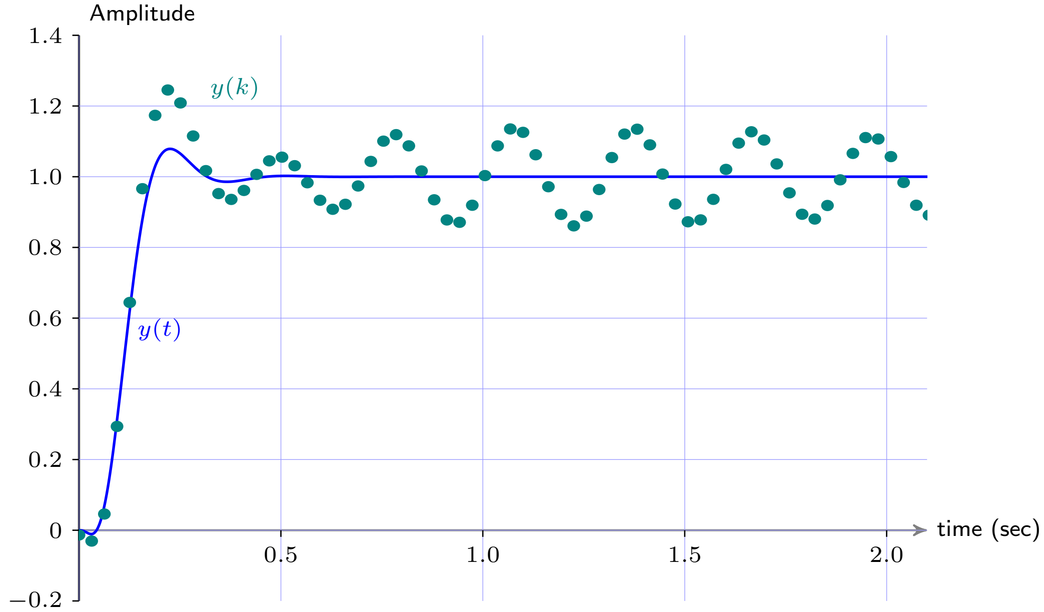


Sensitivity and complementary sensitivity: bilinearly transformed controller



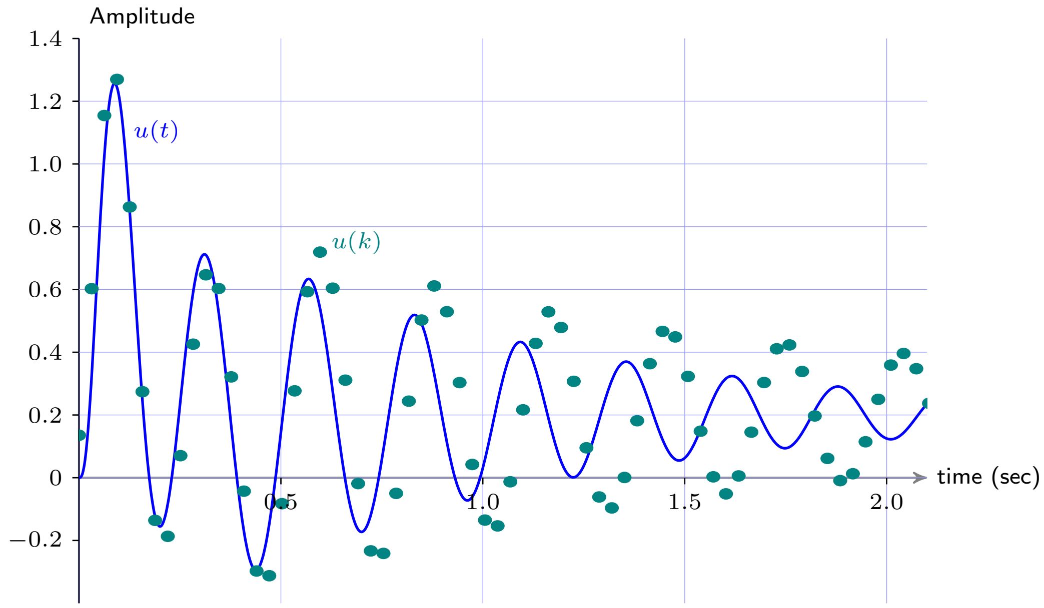
Step response: bilinearly transformed controller

Output



Step response: bilinearly transformed controller

Actuation



Prewarped Tustin transform

Prewarped Tustin transform

Nyquist frequency: 100 radians/second $\Longrightarrow$ $T = \frac{\pi}{100}$.

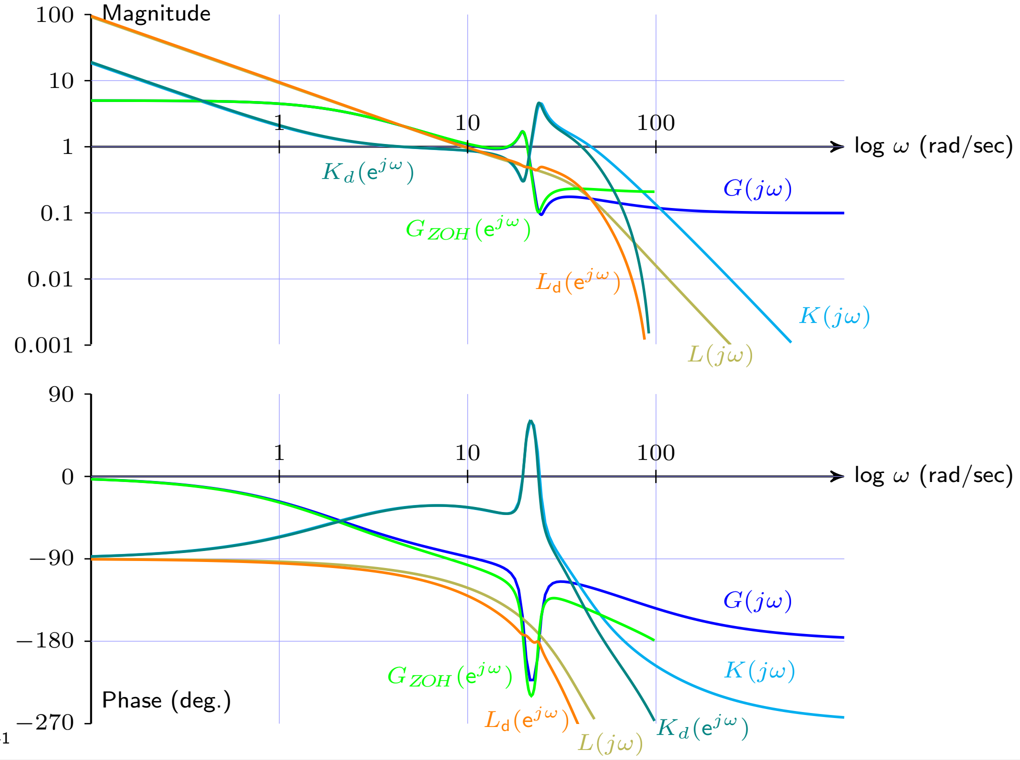
Select the prewarping frequency at $\omega_{\text{pw}} = \omega_m$ (20 radians/sec.).

$$K_{\text{pw}}(z) = K(s)\Big|_{s=\alpha\frac{z-1}{z+1}}$$

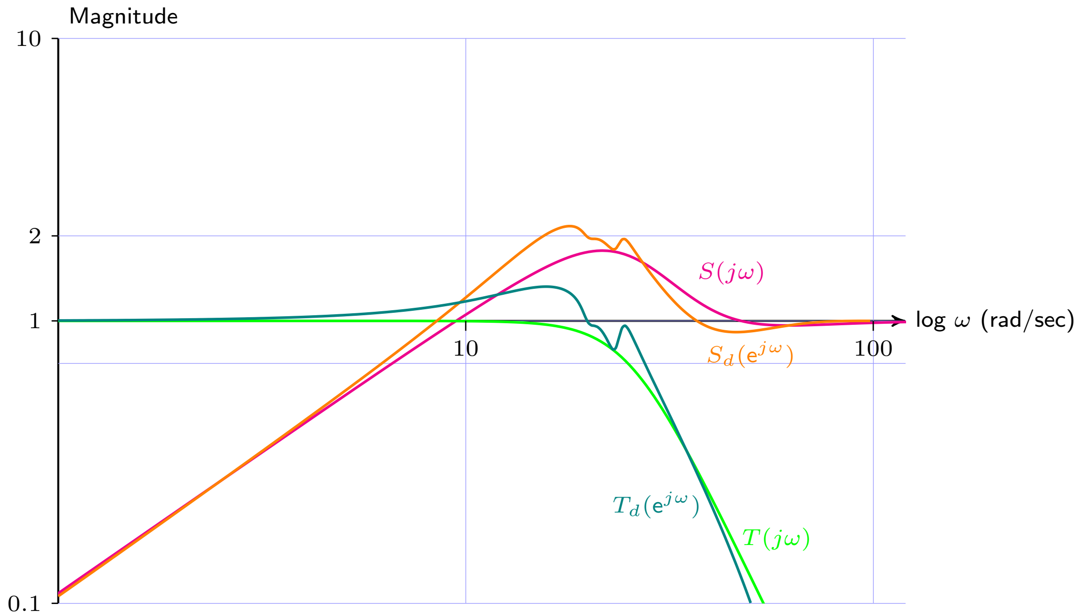
where,

$$\alpha = \frac{\omega_{\text{pw}}}{\tan(\omega_{\text{pw}}T/2)}.$$

Loopshapes: prewarped Tustin controller

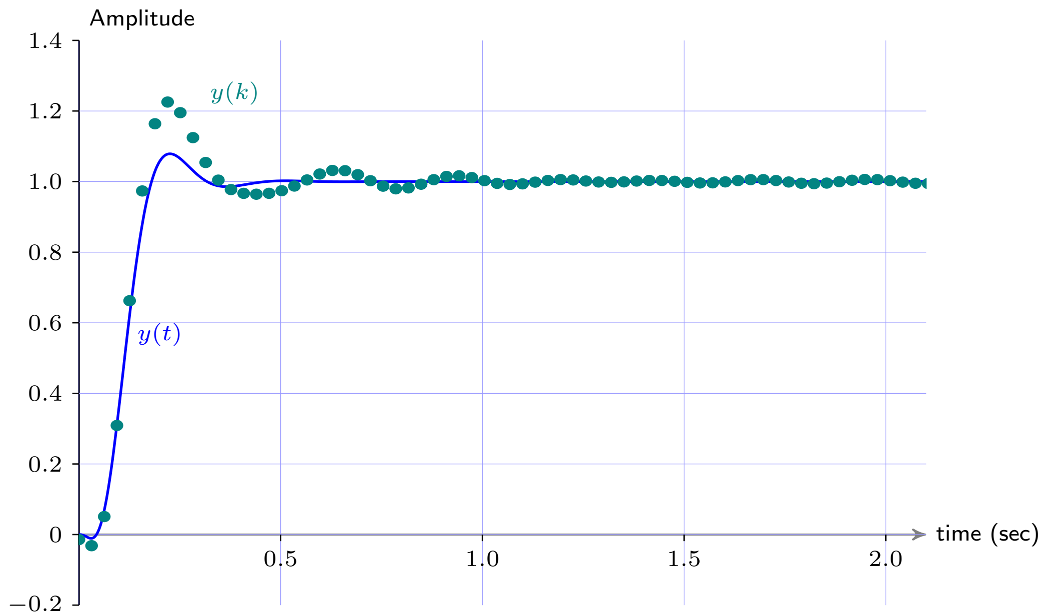


Sensitivity and complementary sensitivity: prewarped Tustin controller



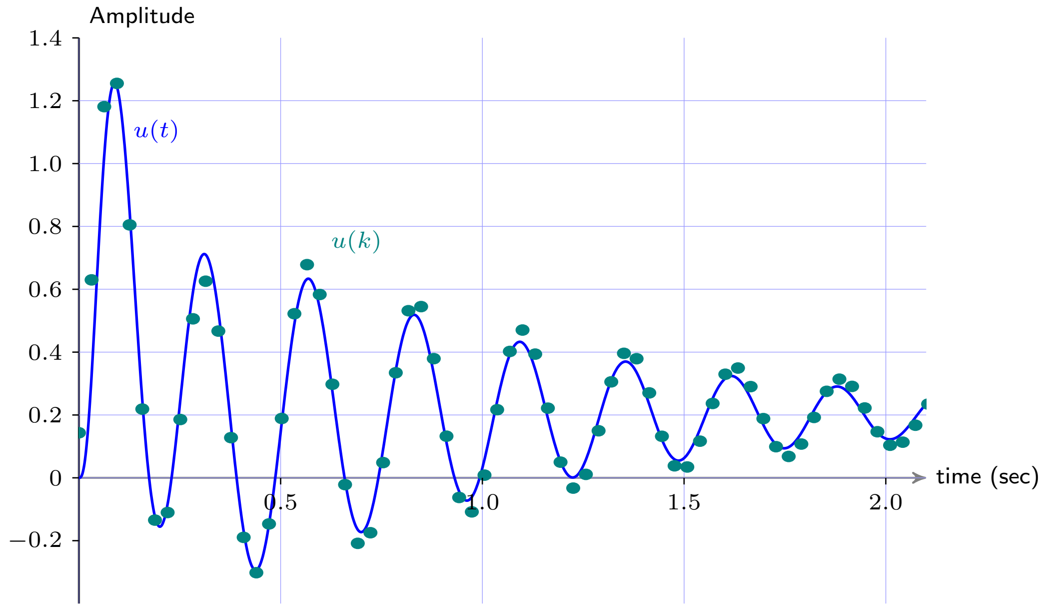
Step response: prewarped Tustin controller

Output



Step response: prewarped Tustin controller

Actuation



Sample rate selection

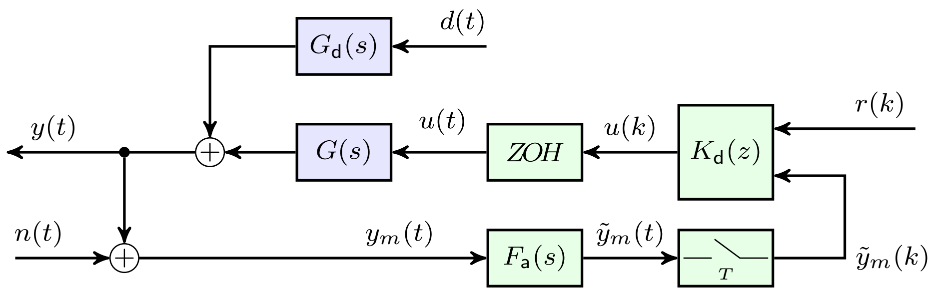
Sample rate selection is critical to digital control design.

Main considerations

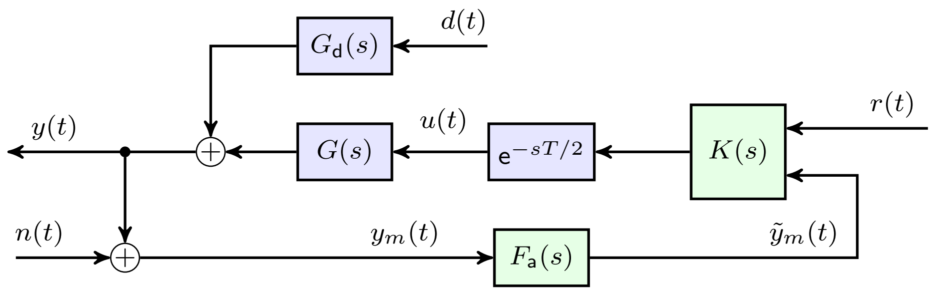
- Sampling/ZOH will (approximately) introduce a delay of $T/2$ seconds.
- Anti-aliasing filters will need to be designed and these will also introduce phase lag.
- The system runs "open-loop" between samples.
- Very fast sampling can introduce additional noise.
- Very fast sampling makes all of the poles appear close to 1. The controller design can become numerically sensitive.

Designing for digital implementation

Sampled-data implementation



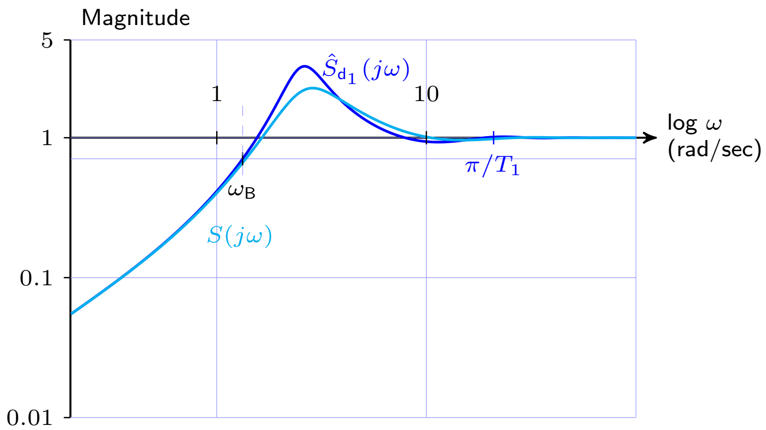
Continuous-time design



Sensitivity function

We want a similar discrete sensitivity function up to the frequency where $|\hat{S}_d(j\omega)|$ returns to 1.

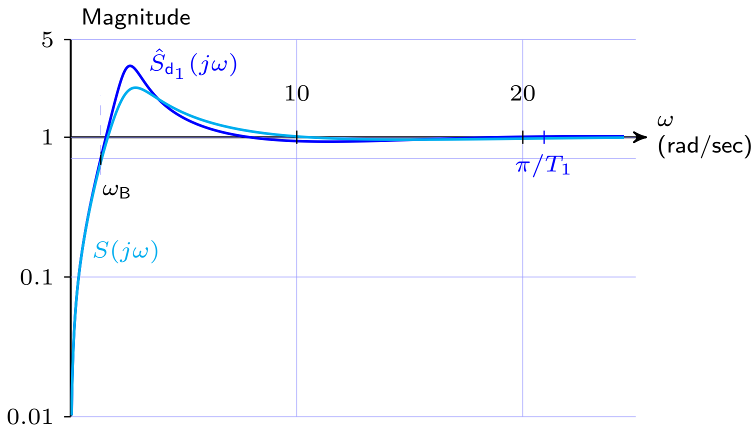
$\hat{S}_d(s) = (I + F_a(s)G(s)e^{-sT_1/2}K(s))^{-1}$ Approximate discrete sensitivity



Sensitivity function

In this example, for $\omega > 20$ rad./sec.,

$$|1 - \hat{S}_{\mathsf{d}}(j\omega)| \ll 1 \quad \Longrightarrow \quad \pi/T = 20 \quad \text{is about the minimum.}$$



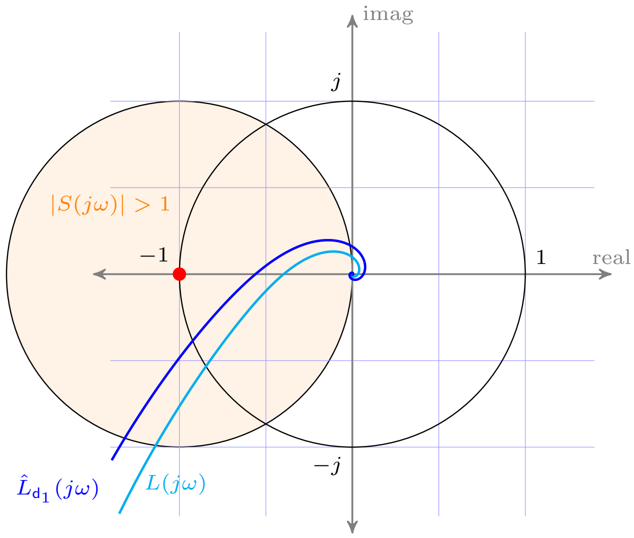
$$\hat{S}_{\mathsf{d}}(s) = (I + F_{\mathsf{a}}(s)G(s)\mathsf{e}^{-sT_1/2}K(s))^{-1}$$

Loop-shaping interpretation

For ω up to where $|F_{\mathsf{a}}(j\omega)G(j\omega)K(j\omega)| < \epsilon$ and remains very small,

we want $\quad F_{\mathsf{a}}(j\omega)G(j\omega)K(j\omega)\mathsf{e}^{-j\omega T_1/2} \approx G_{\mathsf{d}}\left(\mathsf{e}^{j\omega T_1}\right) K_{\mathsf{d}}\left(\mathsf{e}^{j\omega T_1}\right).$

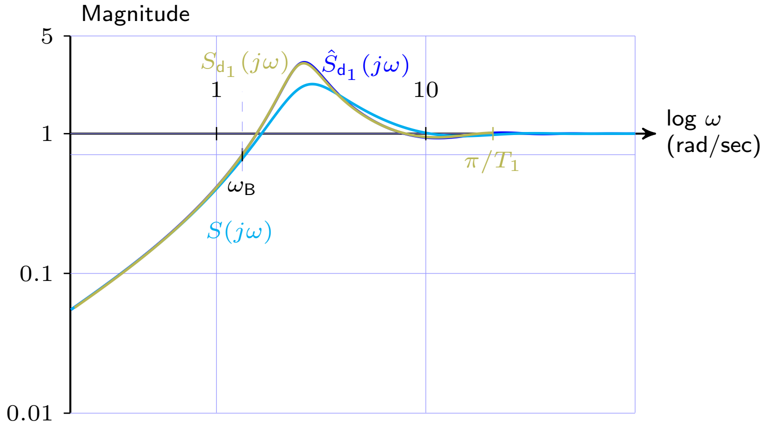
($G_{\mathsf{d}}(z)$ is the ZOH-equivalent of $F_{\mathsf{a}}(s)G(s)$)



Discrete-time sensitivity function

What is the actual discrete sensitivity function,

$$S_d(s) = \frac{1}{(1 + G_d(e^{j\omega})\,K_d(e^{j\omega}))} \quad \text{for} \quad \pi/T_1 = 20.9 \text{ [rad./sec.]?}$$

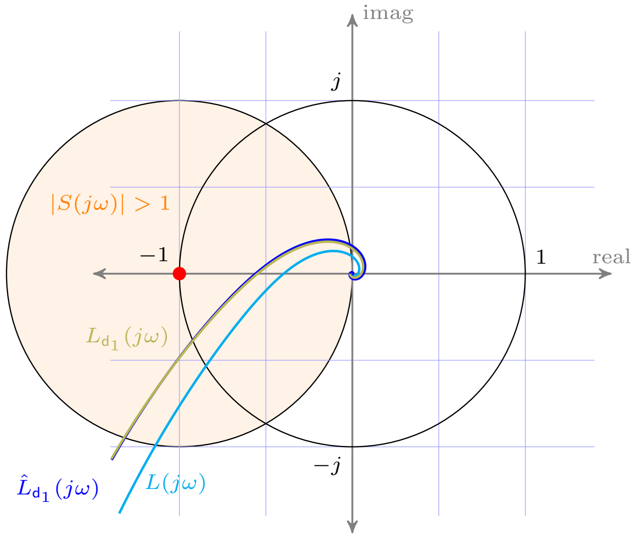


$K_d(z)$ is a prewarped Tustin approximation with $\omega_{pw} = \omega_B$.

Discrete-time loopshape: $L_{d_1}(s) = G_d(s)K_d(s)$, $T_1 = 0.15$ seconds

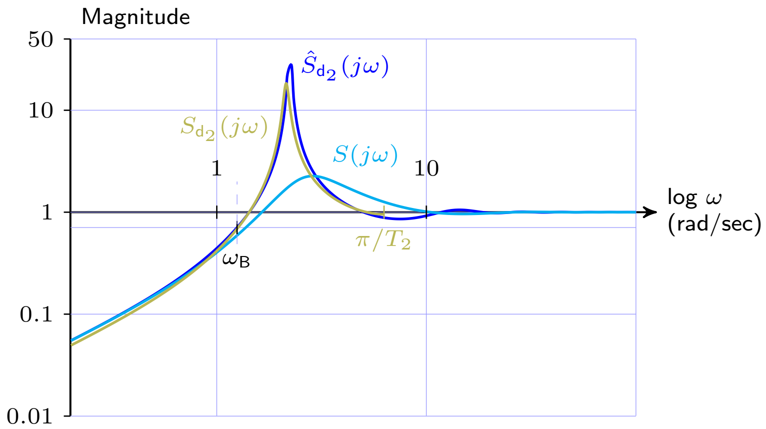
$G_d(s)$ is a ZOH-equivalent transform of $G(s)$.

$K_d(z)$ is a prewarped Tustin approximation with $\omega_{pw} = \omega_B$.



Choosing a slower sample rate.

Choosing $T_2 = 0.5$ seconds $\implies$ $\pi/T_2 = 6.28$ rad./sec.



$K_{\mathsf{d}}(z)$ is a prewarped Tustin approximation with $\omega_{\mathsf{pw}} = \omega_{\mathsf{B}}$.

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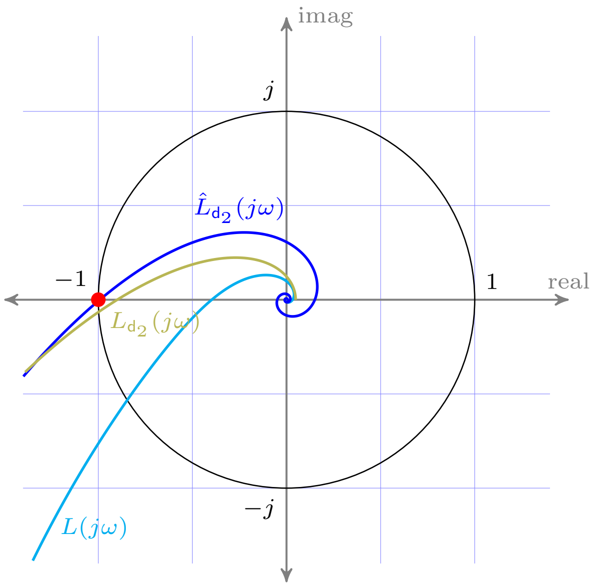

Fast sampling

Fast sampling period: T_f.

Control appropriate (slower) sampling period: T_s
(typically $T_\mathsf{s} = MT_\mathsf{f}$ for integer $M > 1$).