Control Systems 2

Lecture 12: Sampled data control

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8:15 Wednesday 18th May, 2022

Digital implementation of control systems

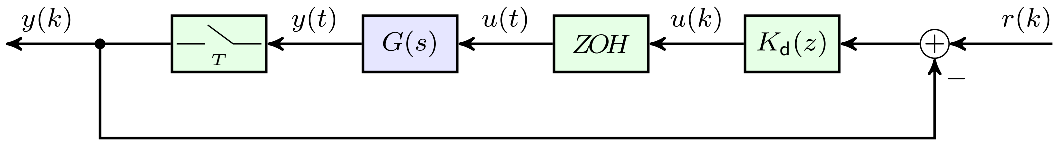
Why digital?

- Easily reprogrammed or modified.
- Complex algorithms (or optimisations) can be implemented.
- Integration with remote systems (via internet).

Why analogue?

- Simple and cheap in mass production.
- Highly reliable.
- Very high frequency operation.
- On-chip integrated systems.

Sampled-data control systems



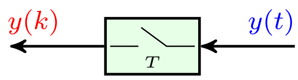
Components:

- Plant: $G(s)$, continuous-time;
- Controller: $K_d(z)$, discrete-time;
- Sampler (A/D converter): $y(k) = y(t)|_{t=kT}$ for $k = 0, 1, 2, \ldots$
- Zero-order hold (D/A converter): $u(t) = u(kT), \quad kT \leq t < kT + T$.

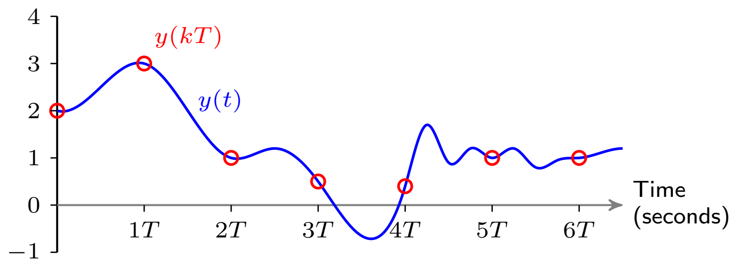
Components: sampler

$y(k) = y(t)\ |_{t=kT}, k = 0, 1, 2, \ldots$

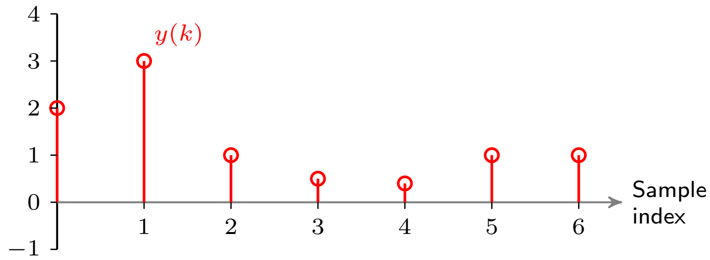
T is the sampling period.



Continuous signal:

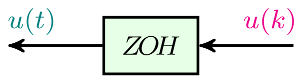


Discrete sequence:

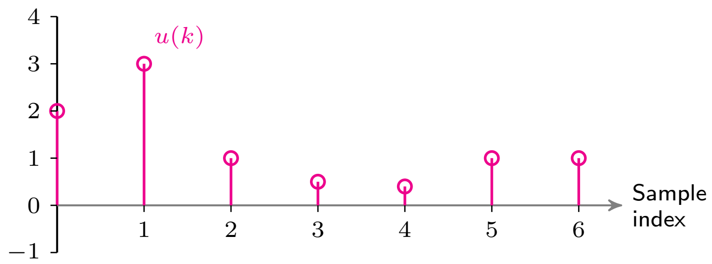


Components: zero-order hold

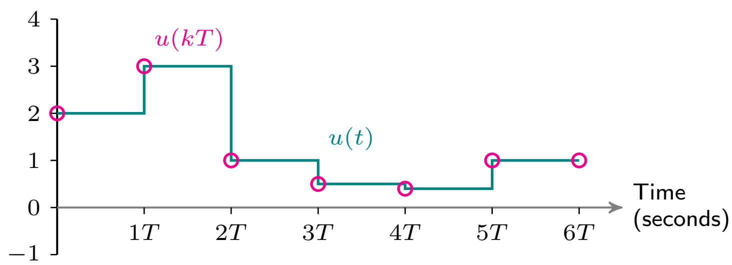
$$u(t) = u(k), \quad \text{for} \quad kT \leq t < kT + T.$$



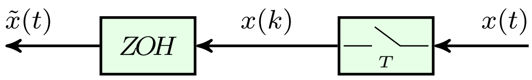
Discrete sequence:

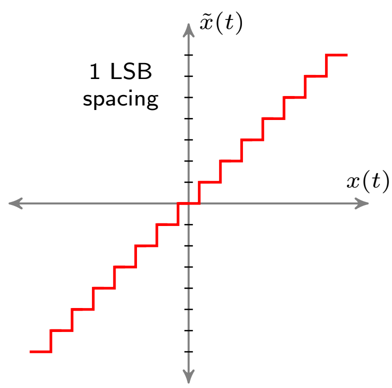


Continuous signal:



Quantisation



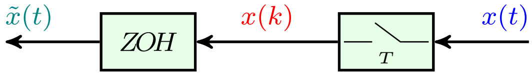


Potential error: $\pm 1/2$ LSB in the best case.

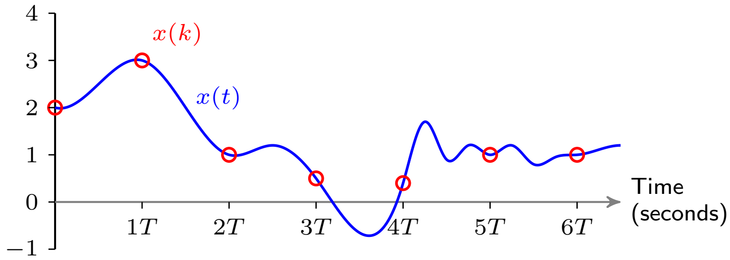
Example: 12 bit A/D and D/A on a ± 10 volt scale

1 LSB $= 0.00488$ volts.

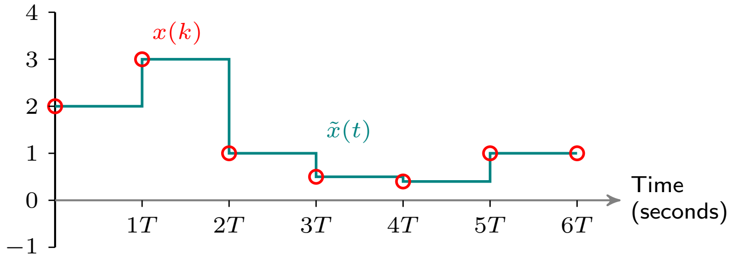
Sampled-data reconstruction



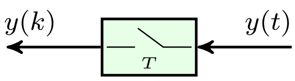
Input signal: $x(t)$



Output signal: $\tilde{x}(t)$



Sampling



Example: single pole signal

Consider $y(t) = \begin{cases} \mathrm{e}^{-at}, & t \geq 0 \\ 0 & t < 0 \end{cases}$ with $a > 0$.

Laplace transform: $y(s) = \dfrac{1}{s+a}$.

Sampled signal: $y(k) = y(t)\Big|_{t=kT} = \mathrm{e}^{-akT} = \left(\mathrm{e}^{-aT}\right)^k$.

Z-transform: $y(z) = \dfrac{z}{z - \mathrm{e}^{-aT}}$.

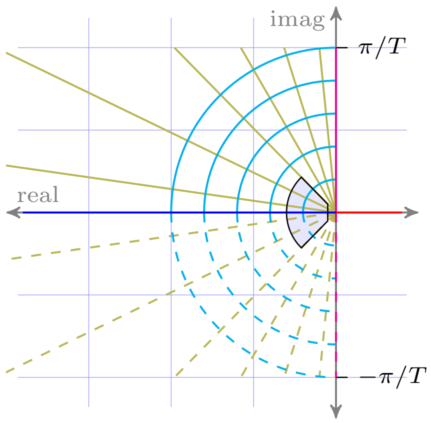
The s-plane pole is at $s_1 = -a$, and the corresponding z-plane pole is at $z_1 = \mathrm{e}^{-aT}$.

Sampling

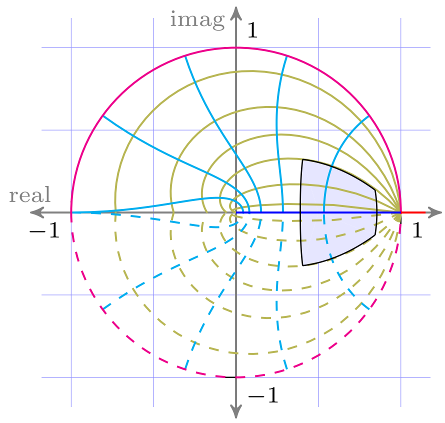
General case:

Sampling maps the s-domain poles to the z-domain via: $z_i = \mathrm{e}^{s_i T}$.
Stability preserving: $\{\mathrm{real}(s_i) < 0\}$ maps to $\{|z_i| < 1\}$.

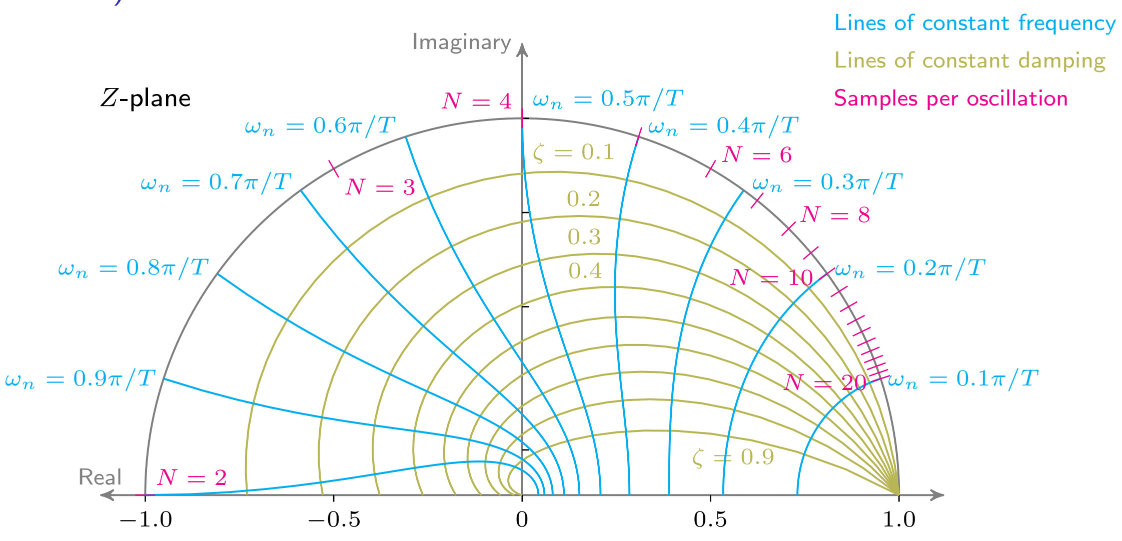
Pole locations under sampling:



$$z = \mathrm{e}^{sT} \longrightarrow$$



Sampling (in detail)



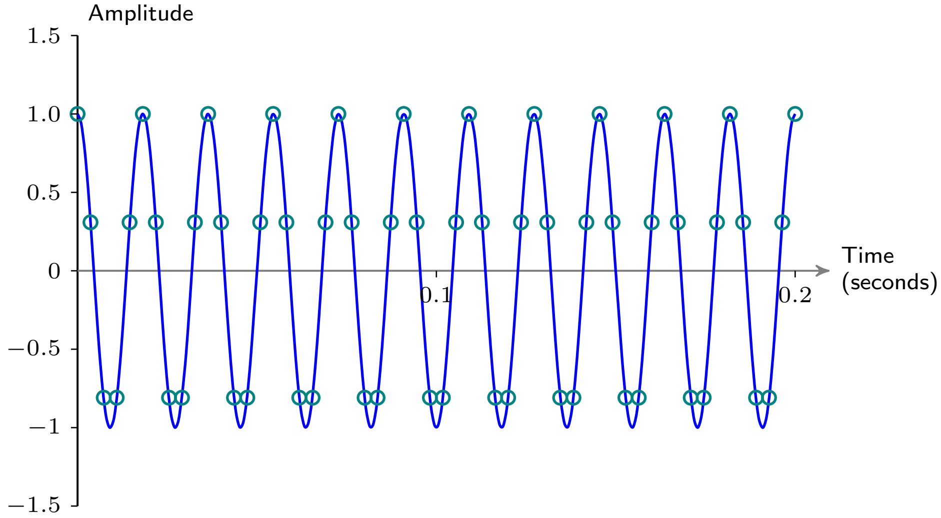
Changing the sampling frequency.

Decreasing T: decrease decay rate ($r \to 1$)
decrease oscillation frequency ($\theta \to 0$)
poles track constant damping curves towards 1

Sampled signals: aliasing

Example:

55 Hz signal sampled at 275 Hz



Sampled signals: aliasing

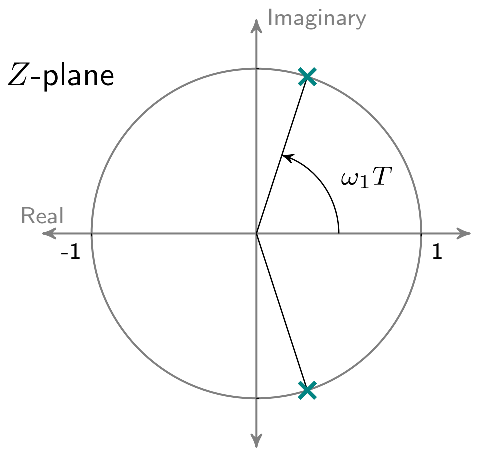
Sampled signal pole mapping: $z_i = e^{s_i T}$

Consider $y(t) = \cos \omega_1 t$

Laplace: $y(s) = \dfrac{s}{s^2 + \omega_1^2}$.

Continuous poles: $s_{1,2} = \pm j\omega_1$

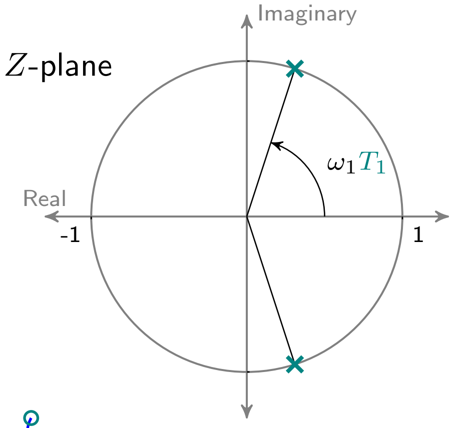
Sampled poles: $z_{1,2} = e^{\pm j\omega_1 T}$

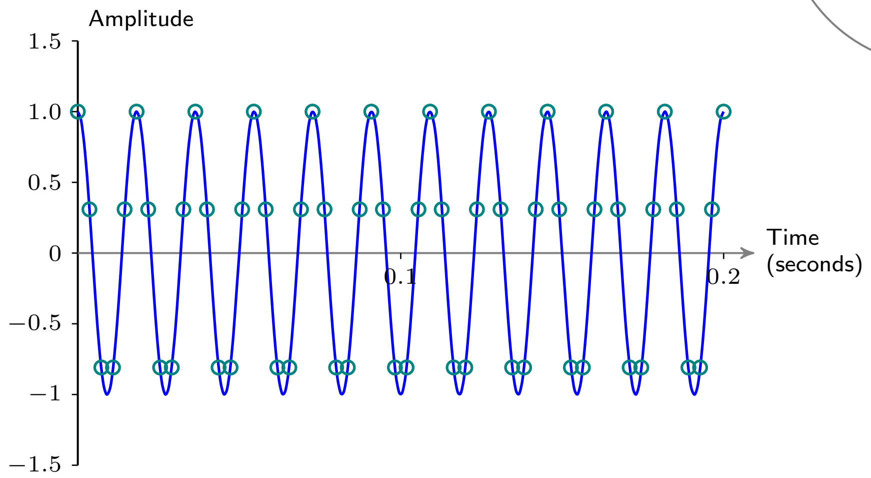


Sampled signals: aliasing

Example:

55 Hz signal sampled at: $1/T_1 = 275$ Hz

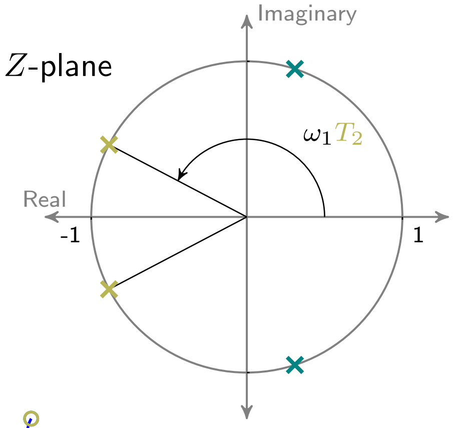


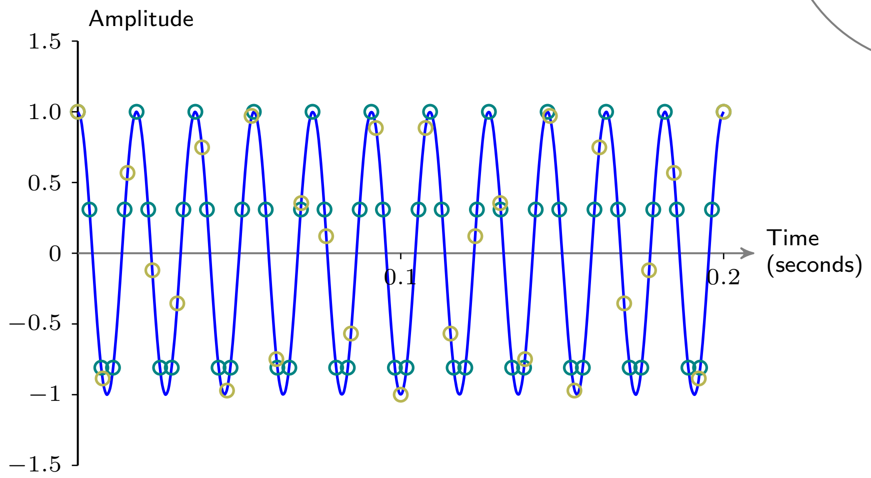


Sampled signals: aliasing

Example:

55 Hz signal sampled at: $1/T_1 = 275$ Hz
$1/T_2 = 130$ Hz

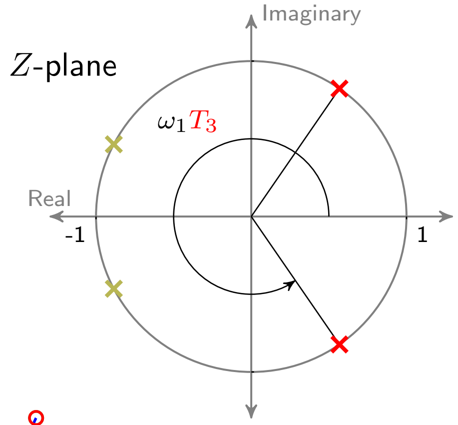


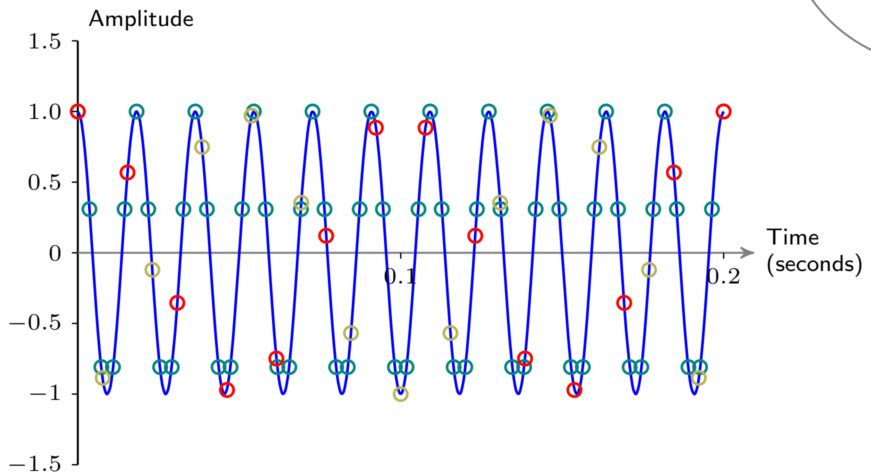


Sampled signals: aliasing

Example:

55 Hz signal sampled at: $1/T_1 = 275$ Hz
$1/T_2 = 130$ Hz
$1/T_3 = 65$ Hz

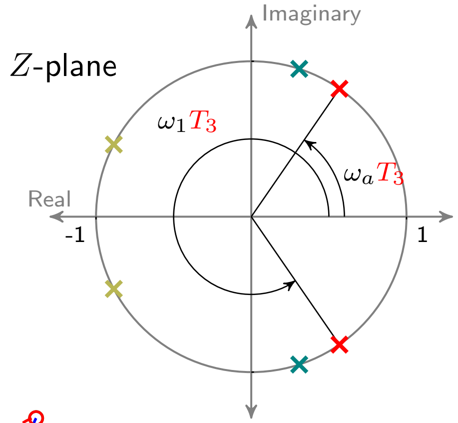


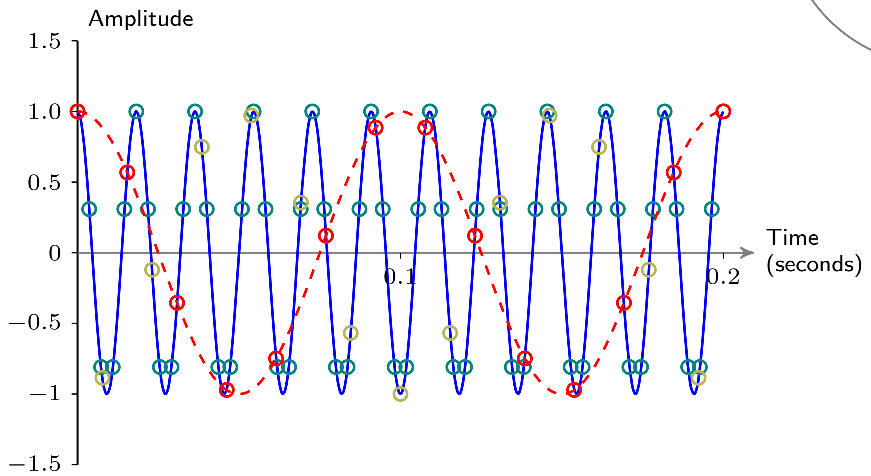


Sampled signals: aliasing

Example:

55 Hz signal sampled at: $1/T_1 = 275$ Hz
$1/T_2 = 130$ Hz
$1/T_3 = 65$ Hz

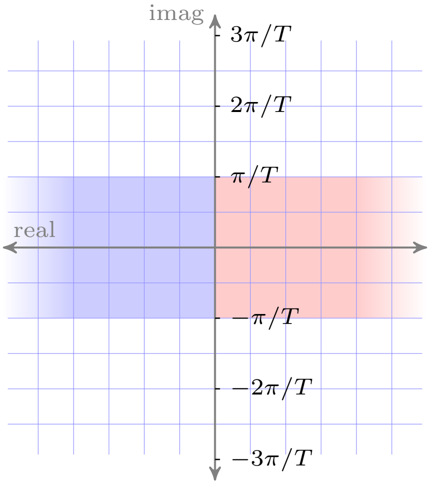




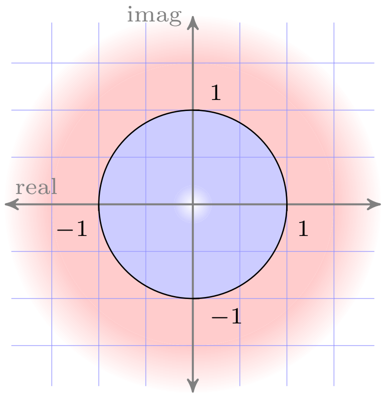
Sampled signals: aliasing

The unit disk can only represent signals of frequency up to 1/2 the sampling frequency. (**Nyquist frequency**).

Maps the horizontal strip from $-j\pi/T$ to $j\pi/T$ onto the z-plane.



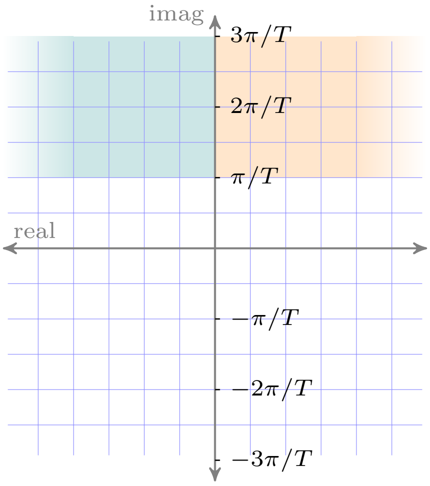
$z = \mathrm{e}^{sT}$
$\longrightarrow$



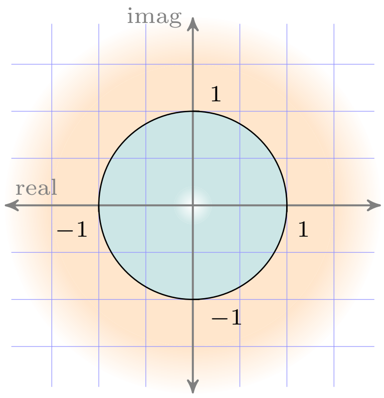
And $\mathrm{real}(s) < 0$ in this strip maps to the inside of the unit disk.

Sampled signals: aliasing

Sampling also maps the next strip (from $j\pi/T$ to $j3\pi/T$) onto the whole z-plane and adds it into the result.



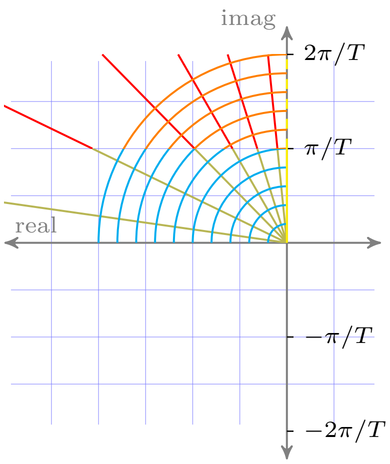
$z = \mathrm{e}^{sT}$
$\longrightarrow$



Also true for all (infinite) $2\pi/T$ wide strips above and below the lowest frequency strip.

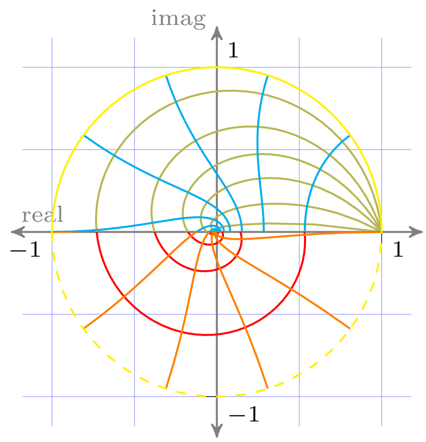
Sampled signals: aliasing

Pole locations under sampling:



$z = \mathrm{e}^{sT}$
$\longrightarrow$

e



Aliased high frequency disturbances are indistinguishable from low frequency disturbances.

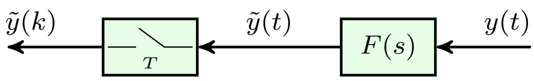
The controller responds at the wrong frequency.

Sampled signals: aliasing

Consequences of aliasing:

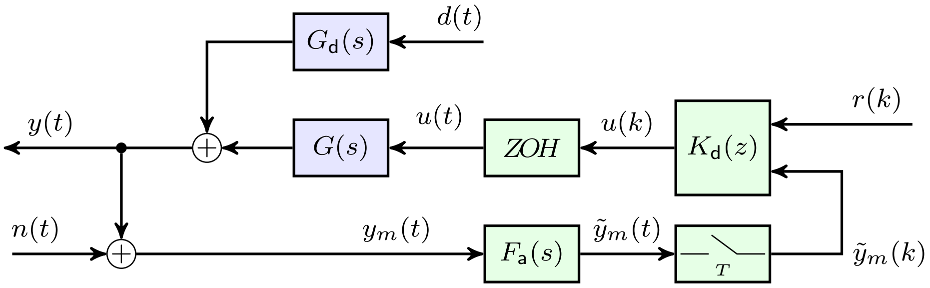
- Ambiguity. Our computer/controller cannot distinguish between frequencies inside the $-\pi/T$ to π/T range and those outside of it.
  - Controller will respond incorrectly to an aliased signal.
  - An aliased signal cannot be reconstructed (signal processing).

Amelioration of the problem:



- Anti-aliasing filter. Low pass, rejecting $|\omega| > \pi/T$.
  - High frequency signals no longer enter loop erroneously.
  - High frequency disturbances/errors are "invisible."

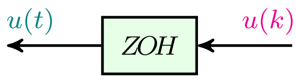
Anti-aliasing filters



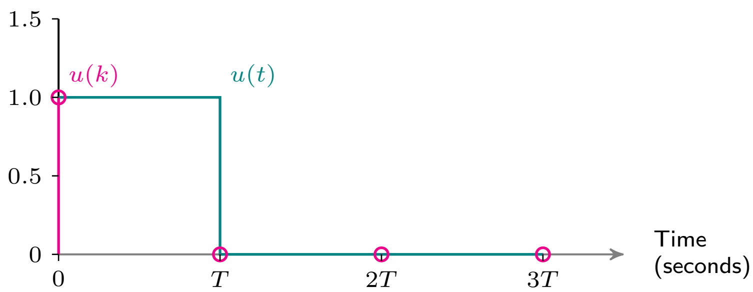
The anti-aliasing filter, $F_a(s)$, will add phase to the loop.

This is **potentially destabilizing!**

ZOH response



Pulse input: $u(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$, gives the output,



Equivalently, the pulse response is:

$$u(t) = \text{step}(t) - \text{step}(t - T), \qquad (\text{step}(t) = \text{unit step function})$$

ZOH response

The discrete-time transfer function is the z-transform of the sampled pulse response.

For a pulse, $u(k)$, the plant input is,

$$u(t) = \text{step}(t) - \text{step}(t - T).$$

The *ZOH* output (in the Laplace domain) is

$$u(s) = \left(1 - \mathrm{e}^{-Ts}\right) \frac{1}{s}.$$

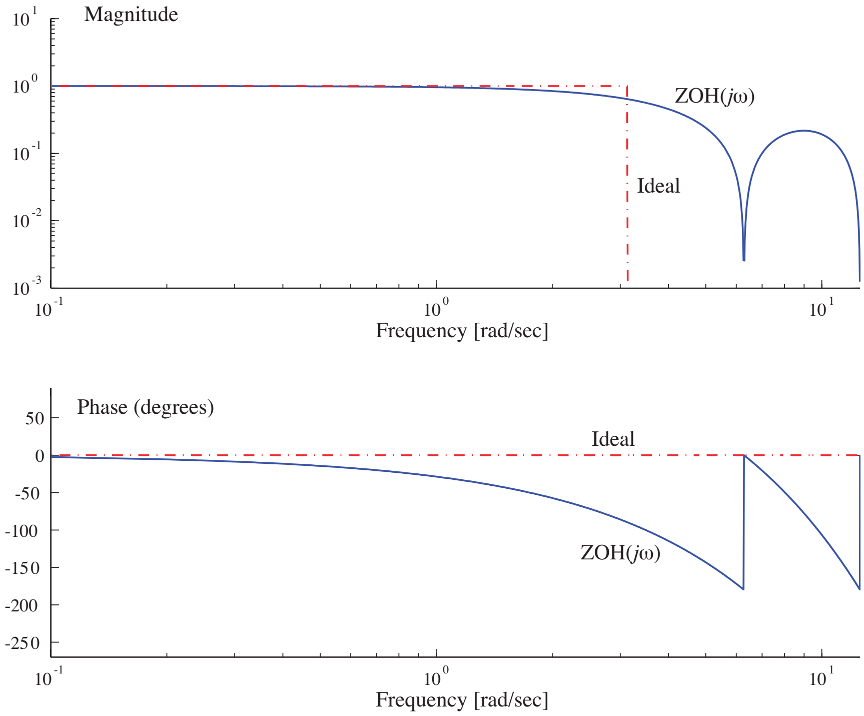
ZOH properties

"Frequency response"

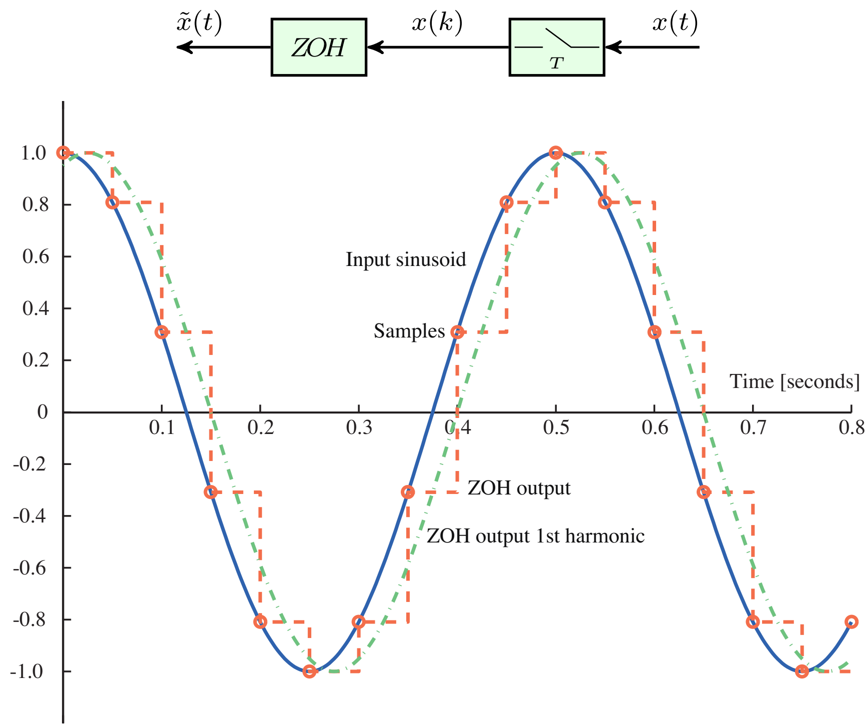
The *ZOH* frequency response (for the fundamental frequency only) is approximated by:

$$\begin{aligned}
ZOH(j\omega) &= \frac{1 - \mathrm{e}^{-j\omega T}}{j\omega} \\
&= \mathrm{e}^{-j\omega T/2} \left(\frac{\mathrm{e}^{j\omega T/2} - \mathrm{e}^{-j\omega T/2}}{2j} \right) \frac{2j}{j\omega} \\
&= T\mathrm{e}^{-j\omega T/2} \frac{\sin(\omega T/2)}{\omega T/2} \\
&= T\mathrm{e}^{-j\omega T/2} \text{sinc}(\omega T/2)
\end{aligned}$$

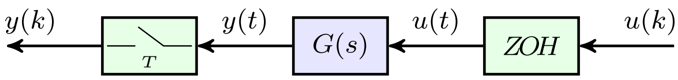
ZOH properties



ZOH properties



Zero-order hold equivalence



Input: $u(k) = \begin{cases} 1 & k = 0, \\ 0 & k \neq 0 \end{cases}, \quad u(t) = \text{step}(t) - \text{step}(t-T).$

Output: $y(s) = \left(1 - \mathrm{e}^{-Ts}\right) \dfrac{G(s)}{s}.$

We now sample this, and take the Z-transform,

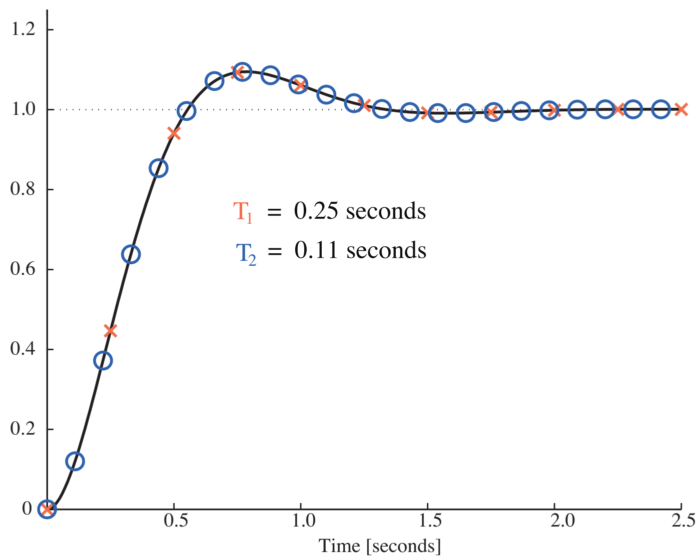
$$\begin{aligned} G_{ZOH}(z) &= \mathcal{Z}\left\{ \left(1 - \mathrm{e}^{-Ts}\right) \frac{G(s)}{s} \right\} \\ &= (1 - z^{-1})\mathcal{Z}\left\{ \frac{G(s)}{s} \right\}. \end{aligned}$$

Easily calculated (`c2d` or `zohequiv` in MATLAB).

Sampling

Sample rate effects:

$z_i = \mathrm{e}^{s_i T}$, changing T changes the pole positions.



Continuous closed-loop step response:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

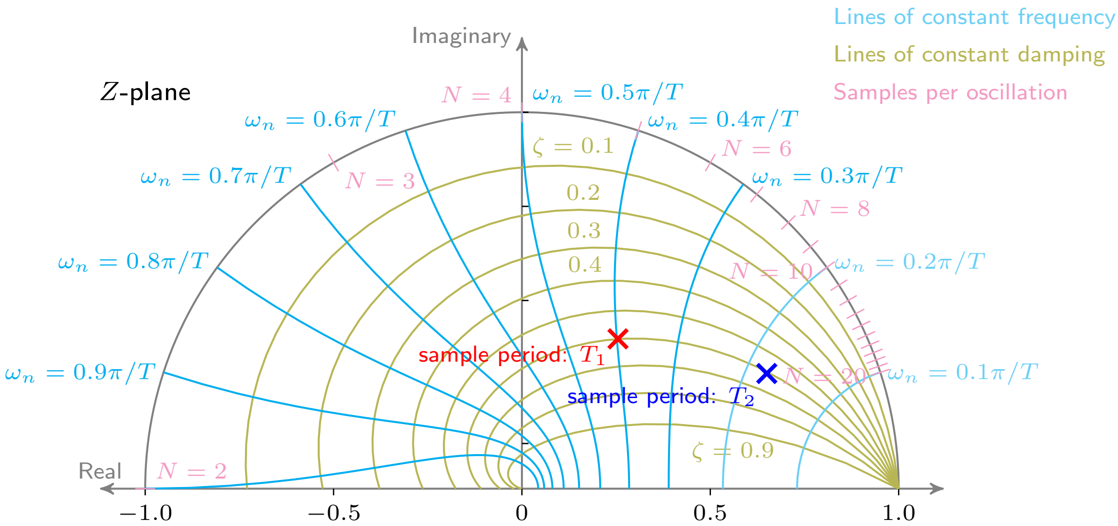
$\zeta = 0.6$,
$\omega_n = 5$ rad./sec.,
$s_{1,2} = -3 \pm 4i$.

Sampling

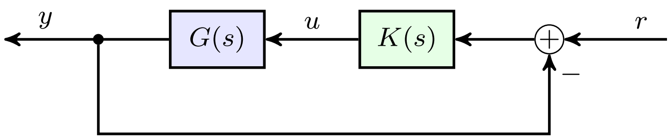
Sample rate effects:

Continuous pole positions: $s_{1,2} = -3 \pm 4i$.

Sample period T_1: $z_{1,2} = 0.255 \pm 0.398i$
Sample period T_2: $z_{1,2} = 0.650 \pm 0.306i$



Continuous time: root-locus analysis of closed-loop stability

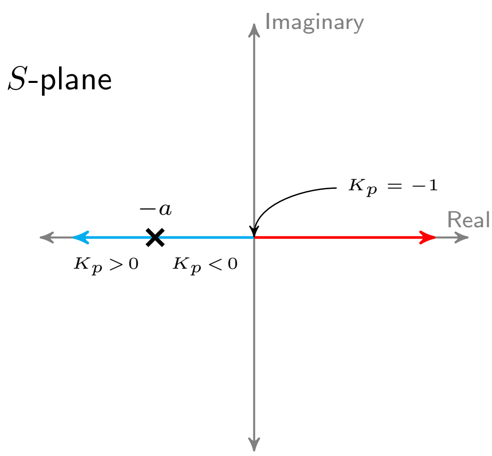


Plant:

$$G(s) = \frac{a}{s+a}, \qquad a > 0.$$

Controller (proportional):

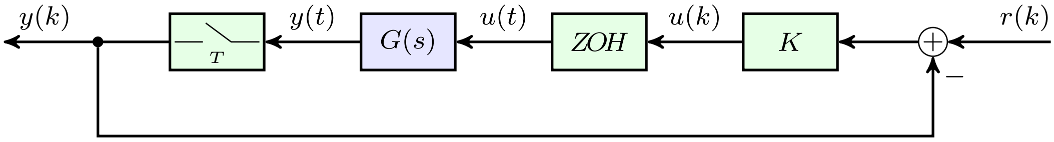
$$K(s) = K_p.$$



$G(s)K(s)$ has one pole and no zeros.

Theoretically stable for $-1 < K_p \leq \infty$.

Discrete time: root-locus analysis of closed-loop stability



$$G_{ZOH}(z) = \frac{1 - e^{-aT}}{z - e^{-aT}}$$

$G_{ZOH}(z)K$ has one pole and no zeros.

Unstable for $K > \dfrac{1 + e^{-aT}}{1 - e^{-aT}}$

The additional phase from the *ZOH* is potentially destabilizing.