Control Systems 2

Lecture 11: MIMO control design: $\mathcal{H}_2$ and $\mathcal{H}_\infty$ methods

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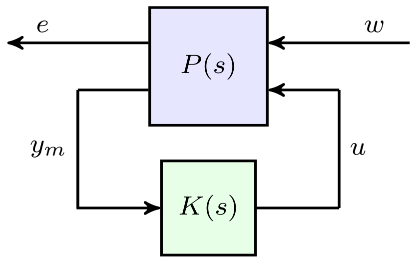
8:15, Wednesday 11th May, 2022

Control synthesis framework



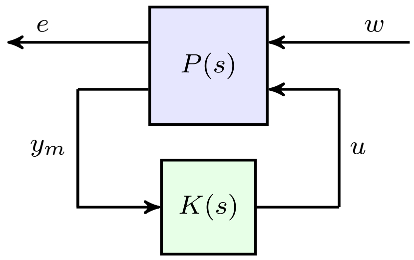
e = performance outputs $\qquad$ w = exogenous inputs

y_m = controller measurements $\qquad$ u = control actuation

Closed-loop transfer function: $e(s) = N(s)w(s)$,

$$N(s) = \mathcal{F}_l\left(P(s), K(s)\right) = P_{11} + P_{12}K\left(I - P_{22}K\right)^{-1}P_{21}$$

Control synthesis framework



$\mathcal{H}_2$ synthesis problem:

$$\underset{K(s)\text{ stabilizing}}{\text{minimize}} \left\|\mathcal{F}_l\left(P(s), K(s)\right)\right\|_{\mathcal{H}_2}$$

$\mathcal{H}_\infty$ synthesis problem:

$$\underset{K(s)\text{ stabilizing}}{\text{minimize}} \left\|\mathcal{F}_l\left(P(s), K(s)\right)\right\|_{\mathcal{H}_\infty}$$

Example

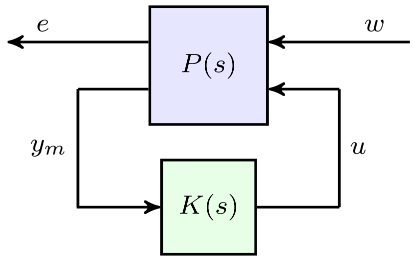
A mixed sensitivity problem (typically $\mathcal{H}_\infty$-norm design):

$$N(s) = \begin{bmatrix} W_{\mathsf{p}}(s)S_{\mathrm{O}}(s) \\ W_{\mathsf{u}}(s)K(s)S_{\mathrm{O}}(s) \\ W_{\mathsf{m}}(s)T_{\mathrm{O}}(s) \end{bmatrix}$$

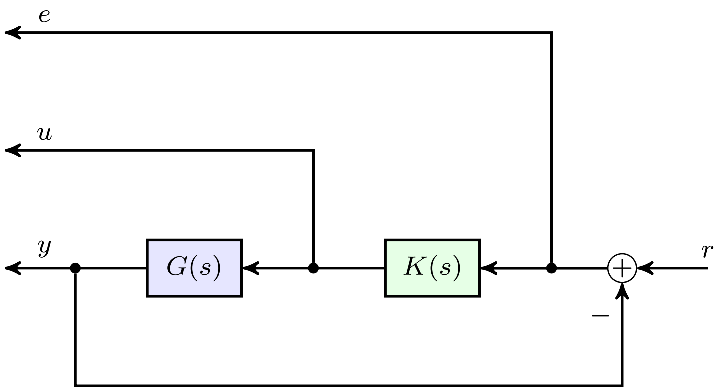
$$\|N(s)\|_{\mathcal{H}_\infty} := \sup_{\omega} \overline{\sigma}\left(N(j\omega)\right)$$

Generalized plant:

$$N(s) = \mathcal{F}_l\left(P(s), K(s)\right)$$

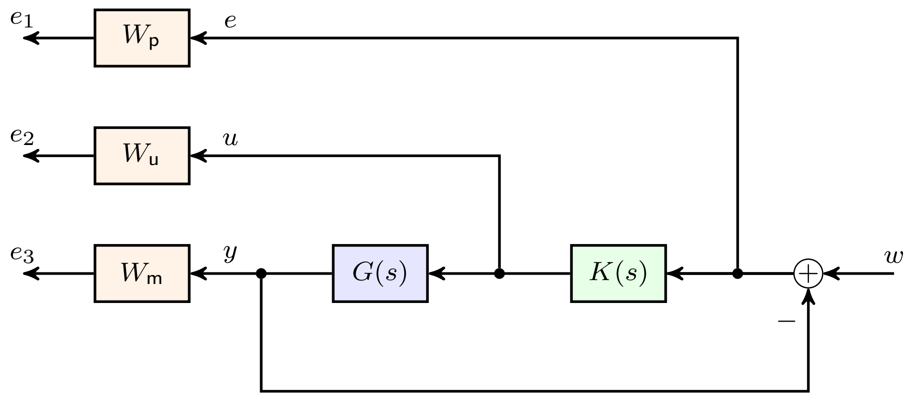


Interconnection structure



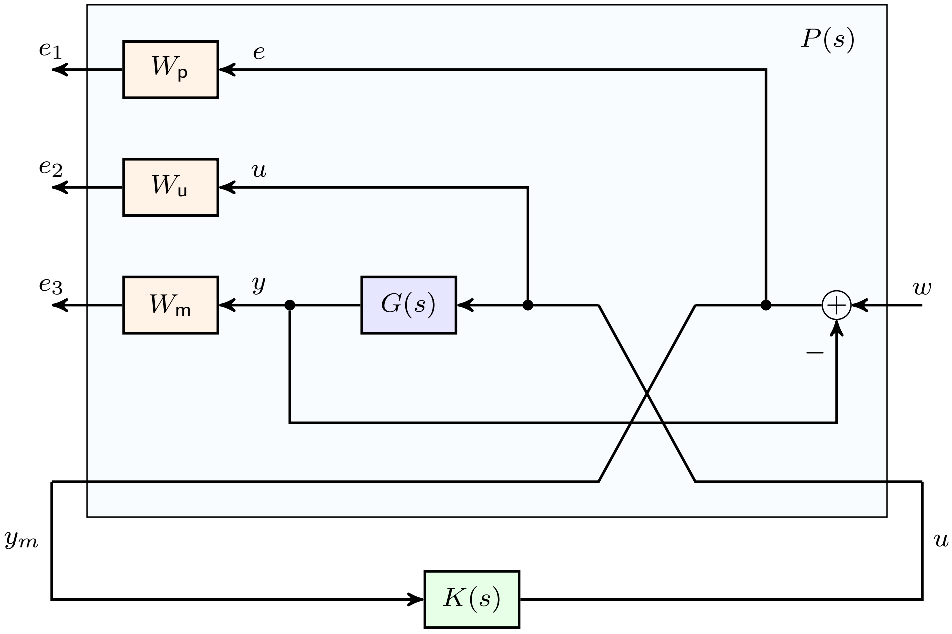
$$N(s) = \begin{bmatrix} W_\mathsf{p}(s)S_\mathrm{o}(s) \\ W_\mathsf{u}(s)K(s)S_\mathrm{o}(s) \\ W_\mathsf{m}(s)T_\mathrm{o}(s) \end{bmatrix}$$

Interconnection structure



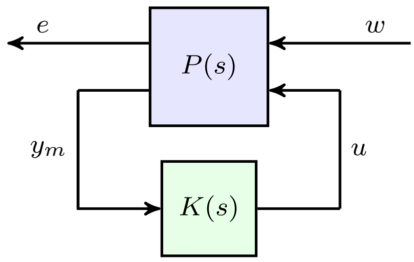
$$N(s) = \begin{bmatrix} W_\mathsf{p}(s)S_\mathrm{o}(s) \\ W_\mathsf{u}(s)K(s)S_\mathrm{o}(s) \\ W_\mathsf{m}(s)T_\mathrm{o}(s) \end{bmatrix}$$

Interconnection structure



$N(s) = \mathcal{F}_l\left(P(s), K(s)\right)$

$\mathcal{H}_2$ synthesis



$$
\begin{aligned}
\|N(s)\|_{\mathcal{H}_2}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}\left(N(j\omega)^* N(j\omega)\right)\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \|N(j\omega)\|_F^2 \, d\omega \\
&= \int_0^{\infty} \operatorname{trace}\left(n(\tau)^T n(\tau)\right)\, d\tau
\end{aligned}
$$

($n(\tau)$ is the impulse response of $N(s)$)

$\mathcal{H}_2$ synthesis

$\|N(s)\|_{\mathcal{H}_2} < 1$ implies:

- If $w(t) = \delta(t)$, then $\|e(t)\|_2 < 1$.
- If $\|w(t)\|_2 < 1$, then $\max_t |e(t)| < 1$.
- If $w(t)$ is unit variance white noise, the $\mathrm{var}(e(t)) < 1$.

For state-space representations:

$$\begin{aligned}
\|N(s)\|_{\mathcal{H}_2}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}\left(N(j\omega)^* N(j\omega)\right)\, d\omega \\
&= \int_0^{\infty} \mathrm{trace}\left(B^T \mathrm{e}^{A^T\tau} C^T C \mathrm{e}^{A\tau} B\right)\, d\tau \\
&= \mathrm{trace}(B^T W_o B) = \mathrm{trace}(C W_c C^T)
\end{aligned}$$

LQG control

Plant $G(s)$:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + w_d \\
y_m(t) &= Cx(t) + w_n
\end{aligned}$$

Process disturbance and measurement noise covariances:

$$E\left\{\begin{bmatrix} w_d(t) \\ w_n(t) \end{bmatrix} \begin{bmatrix} w_d(\tau)^T & w_n(\tau)^T \end{bmatrix}\right\} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \delta(t-\tau)$$

LQG control design problem:

Find $u(s) = K(s) y_m(s)$ to minimize,

$$J = E\left\{\lim_{T\longrightarrow\infty} \frac{1}{T}\int_0^T \left(x(t)^T Q x(t) + u(t)^T R u(t)\right) dt\right\},$$

with $Q = Q^T \geq 0$ and $R = R^T > 0$.

LQG control

Generalized plant $P(s)$:

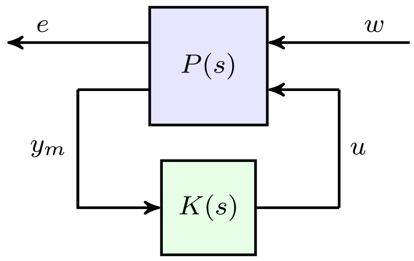
$$e = \begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} w_d \\ w_n \end{bmatrix} = \begin{bmatrix} W^{1/2} & 0 \\ 0 & V^{1/2} \end{bmatrix} w$$

With $w(t)$ unit variance white noise.

LQG cost function:

$$J = E\left\{ \lim_{T \longrightarrow \infty} \frac{1}{T} \int_0^T e(t)^T e(t)\, dt \right\}$$
$$= \|\mathcal{F}_l\left(P(s), K(s)\right)\|_{\mathcal{H}_2}^2$$

LQG control

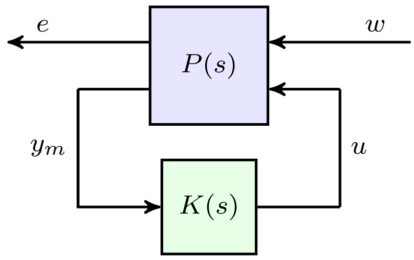


$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{c|ccc} A & W^{1/2} & 0 & B \\ \hline Q^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & R^{1/2} \\ C & 0 & V^{1/2} & 0 \end{array}\right]$$

$\mathcal{H}_2$ synthesis problem:

$$\underset{K(s)\ \text{stabilizing}}{\text{minimize}} \|\mathcal{F}_l\left(P(s), K(s)\right)\|_{\mathcal{H}_2}$$

$\mathcal{H}_\infty$ synthesis



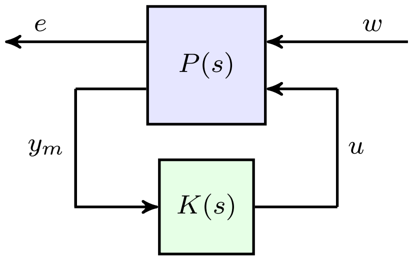
$$\|N(s)\|_{\mathcal{H}_\infty} := \sup_{w(s)\neq 0} \frac{\|e(s)\|_2}{\|w(s)\|_2} \qquad \text{(induced norm with the space)}$$

$$= \sup_{w(s)\neq 0} \frac{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e(j\omega)^T e(j\omega) d\omega\right)^{1/2}}{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} w(j\omega)^T w(j\omega) d\omega\right)^{1/2}}$$

$$= \max_{\omega} \overline{\sigma}\left(N(j\omega)\right) \quad \text{(for } N(s) \text{ stable)}$$

$\mathcal{H}_\infty$ synthesis

$$P(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_e & D_{ew} & D_{eu} \\ C_y & D_{yw} & D_{yu} \end{array}\right]$$

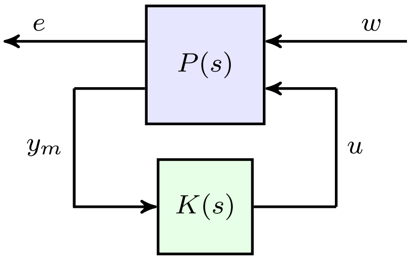


Assumptions on $P(s)$:

1. (A, B_u, C_y) are stabilizable and detectable.

$\mathcal{H}_\infty$ synthesis

$$P(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_e & D_{ew} & D_{eu} \\ C_y & D_{yw} & D_{yu} \end{array}\right]$$

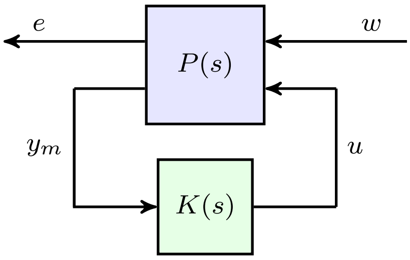


Assumptions on $P(s)$:

1. $(A,\ B_u,\ C_y)$ are stabilizable and detectable.
2. D_{eu} and D_{yw} are full rank.

$\mathcal{H}_\infty$ synthesis

$$P(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_e & D_{ew} & D_{eu} \\ C_y & D_{yw} & D_{yu} \end{array}\right]$$



Assumptions on $P(s)$:

1. $(A,\ B_u,\ C_y)$ are stabilizable and detectable.
2. D_{eu} and D_{yw} are full rank.
3. $\begin{bmatrix} A - j\omega I & B_u \\ C_e & D_{eu} \end{bmatrix}$ has full column rank for all ω.
4. $\begin{bmatrix} A - j\omega I & B_w \\ C_y & D_{yw} \end{bmatrix}$ has full row rank for all ω.

$\mathcal{H}_\infty$ synthesis

Suboptimal problem:

Given $\gamma > 0$, find a stabilising $K(s)$ such that,

$$\|\mathcal{F}_l\left(P(s), K(s)\right)\|_{\mathcal{H}_\infty} \leq \gamma \qquad \text{(feasible solution)}$$

$\mathcal{H}_\infty$ synthesis

Find the smallest $\gamma > 0$, for which there exists a feasible $K(s)$ satisfying,

$$\|\mathcal{F}_l\left(P(s), K(s)\right)\|_{\mathcal{H}_\infty} \leq \gamma$$

MATLAB command:

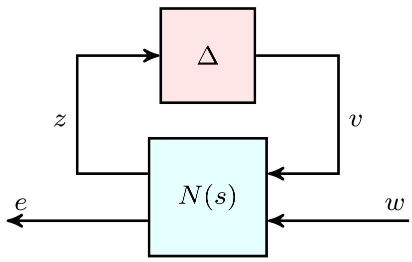
```
>>  [K,N,gamma] = hinfsyn(P,ny,nu)
```


Robust performance

Analysis:

$$\text{RP} \iff \mu_{\tilde{\mathbf{\Delta}}}(N(j\omega)) < 1, \quad \text{for all } \omega$$

$$\tilde{\mathbf{\Delta}} = \text{diag}(\mathbf{\Delta}, \Delta_p)$$

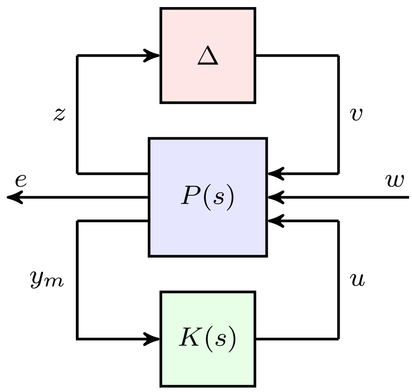


Synthesis:

$$\min_{K(s) \text{ stabilizing}} \gamma$$

subject to:

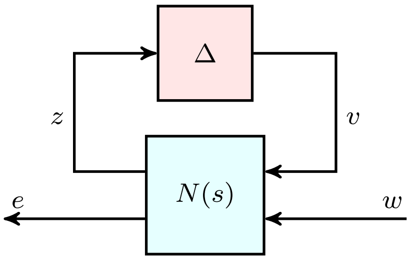
$$\mu_{\tilde{\mathbf{\Delta}}}(\mathcal{F}_l\left(P(j\omega), K(j\omega)\right)) < \gamma, \quad \text{for all } \omega$$

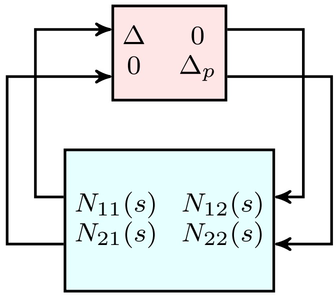


Robust performance

As an equivalent robust stability problem

Robust performance $\iff$ Robust stability





RS : $\mu_{\mathbf{\Delta}}(N_{11}(j\omega)) < 1$, for all ω

RP : $\|\mathcal{F}_u\left(N(s), \Delta\right)\|_{\mathcal{H}_\infty} \le 1$
for all $\Delta \in \mathbf{\Delta}$

$\mu_{\tilde{\mathbf{\Delta}}}(N(j\omega)) < 1$, for all ω

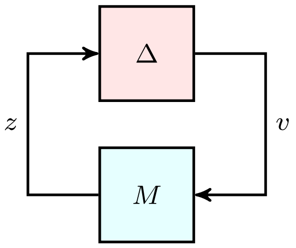
$$\tilde{\mathbf{\Delta}} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_p \end{bmatrix}$$

$\Delta \in \mathbf{\Delta}, \Delta_p \in \mathcal{C}^{\dim(w)\times\dim(e)}$

μ upper bound: D scaling

If an invertible matrix D commutes with all $\Delta \in \mathbf{\Delta}$, then,

$$\mu_{\mathbf{\Delta}}(M) \ = \ \mu_{\mathbf{\Delta}}(DMD^{-1})$$



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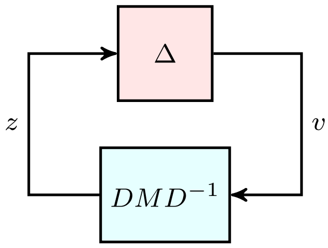


A μ upper bound (for robust performance):

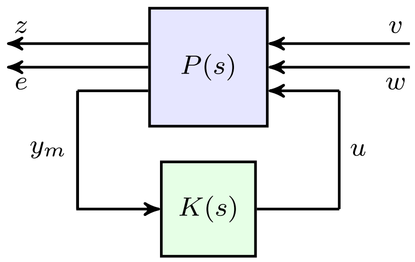
$$\mu_{\tilde{\mathbf{\Delta}}}(M) \le \inf_{D\in\mathcal{D}} \overline{\sigma}\left(DMD^{-1}\right)$$

Notation: $\mathcal{D} = \{\, D \mid D\Delta D^{-1} \in \mathbf{\Delta} \text{ for all } \Delta \in \mathbf{\Delta}\,\}$

D-K iteration:

$\mathcal{H}_\infty$ synthesis:

$$\min_{K(s)\text{stabilizing}} \|\mathcal{F}_l\left(P(s), K(s)\right)\|_{\mathcal{H}_\infty}$$



This gives an upper bound on the robust performance:

$$N(s) = \mathcal{F}_l\left(P(s), K(s)\right) \quad \text{and} \quad \text{RP} \iff \mu_{\tilde{\mathbf{\Delta}}}(N(s)) \leq 1.$$

$$\sup_\omega \mu_{\tilde{\mathbf{\Delta}}}(N(j\omega)) \leq \sup_\omega \overline{\sigma}\left(N(j\omega)\right) = \|N(s)\|_{\mathcal{H}_\infty}$$

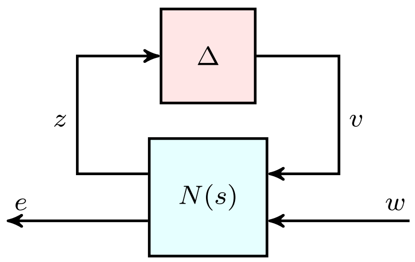
If $\|N(s)\|_{\mathcal{H}_\infty} = \gamma \leq 1$ then $\mu_{\tilde{\mathbf{\Delta}}}(N(j\omega)) \leq 1,$ for all ω (RP)

D-K iteration

μ test for robust performance

$N(s) = \mathcal{F}_l\left(P(s), K(s)\right)$

Is $\mu_{\tilde{\mathbf{\Delta}}}(N(j\omega)) \leq 1$ for all ω?



μ upper bound calculation

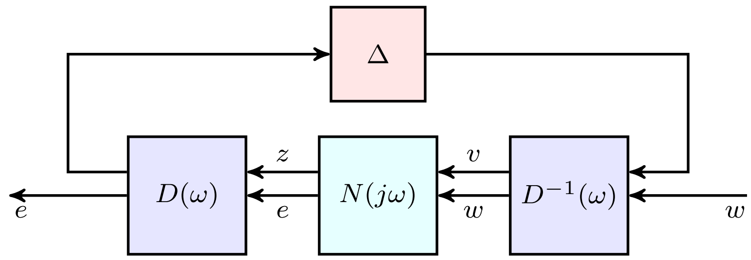
As $\mu_{\tilde{\mathbf{\Delta}}}(N(j\omega)) \leq \inf_{D(\omega)\in\mathcal{D}} \overline{\sigma}\left(D(\omega)N(j\omega)D^{-1}(\omega)\right),$

If for every ω there exists $D(\omega) \in \mathcal{D}$ such that,

$$\overline{\sigma}\left(D(\omega)N(j\omega)D^{-1}(\omega)\right) \leq 1, \qquad \text{then RP is satisfied.}$$

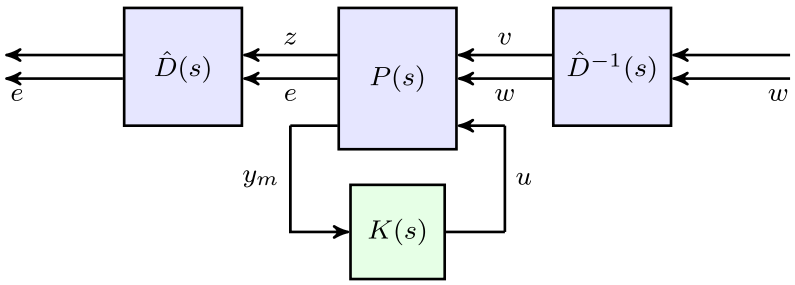
Iterative design

μ upper bound:



Scaled $\mathcal{H}_\infty$ design problem:

Find a stable, invertible $\hat{D}(s)$ such that $\hat{D}(j\omega) \approx D(j\omega)$.

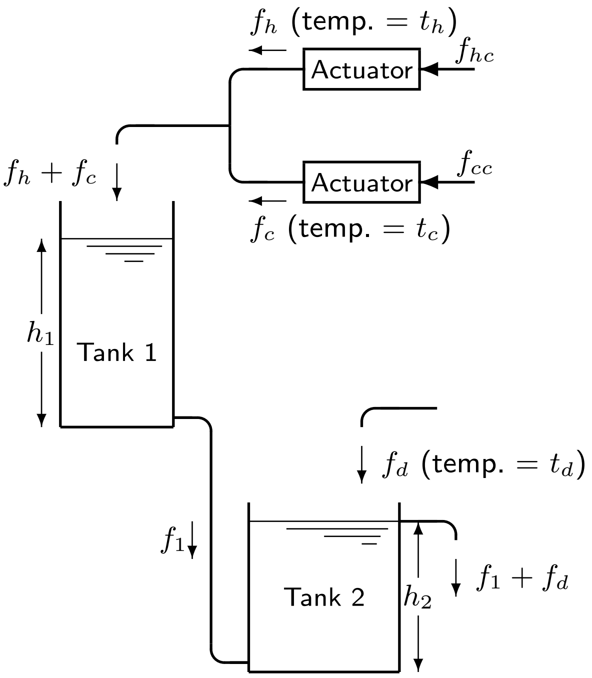


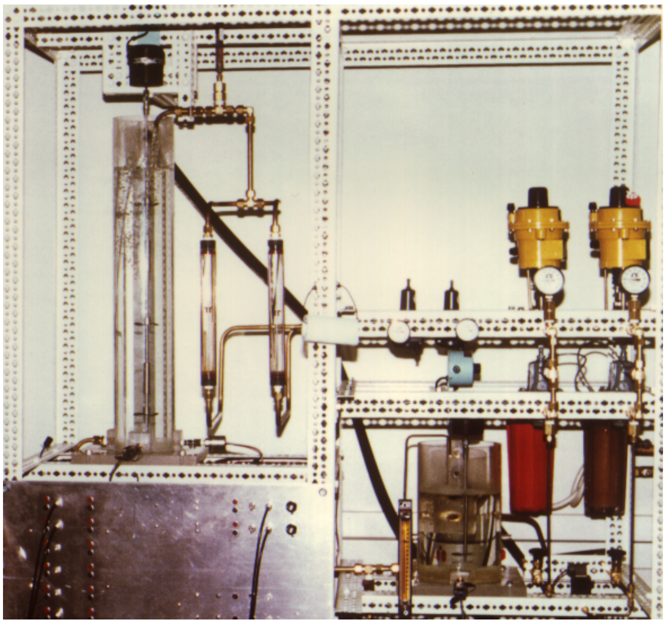
Robust performance sysnthesis:

D-K iteration

1. Initialize procedure with $K_0(s)$ (nominal $\mathcal{H}_\infty$ controller)
2. Calculate closed-loop: $N(s) = \mathcal{F}_l\left(P(s), K_0(s)\right)$
3. Calculate upper bound D scalings (for a grid, ω):
   $$\inf_{D(\omega) \in \mathcal{D}} \overline{\sigma}\left(D(\omega)N(j\omega)D(\omega)^{-1}\right)$$
4. Approximate $D(\omega)$ with stable, invertible system $\hat{D}(s)$, such that $|\hat{D}(j\omega)| \approx D(\omega)$.
5. Design $\mathcal{H}_\infty$ controller for $\hat{D}(s)P(s)\hat{D}^{-1}(s)$.
6. If $\mu_{\tilde{\mathbf{\Delta}}}(N(j\omega)) \geq 1$, for any ω, go to step 3.

D-K iteration: Tank experiment process control example





Plant model (Tank 1)

f_1	flow out of Tank 1, $f_1 \in [0,1]$	
h_1	water height in Tank 1, $h_1 \in [0.15, 0.75]$	$\longleftarrow$ output #1
A_1	cross-sectional area of Tank 1, $A_1 = 91.4$	
t_1	temperature in Tank 1, $t_1 \in [0,1]$	$\longleftarrow$ output #2
f_h	hot water flow rate, $f_h \in [0,1]$	$\longleftarrow$ input #1
f_c	cold water flow rate, $f_c \in [0,1]$	$\longleftarrow$ input #2
t_h	hot supply temperature, $t_h = 1.0$	
t_c	cold supply temperature, $t_c = 0.0$	
α	height/flow model gain, $\alpha = 1.34$	
β	height/flow model bias, $\beta = 0.6$	
E_1	"energy" variable, defined as $E_1 = h_1 t_1$	

$$
\begin{aligned}
\dot{f}_1 &= \frac{-1}{A_1\alpha} f_1 + \frac{1}{A_1\alpha} f_h + \frac{1}{A_1\alpha} f_c, \\
h_1 &= \alpha f_1 - \beta, \\
\dot{E}_1 &= \frac{-1}{A_1\alpha}\left(1 + \frac{\beta}{h_1}\right) E_1 + \frac{t_h}{A_1} f_h + \frac{t_c}{A_1} f_c, \\
t_1 &= \frac{E_1}{h_1}.
\end{aligned}
$$

Nominal plant

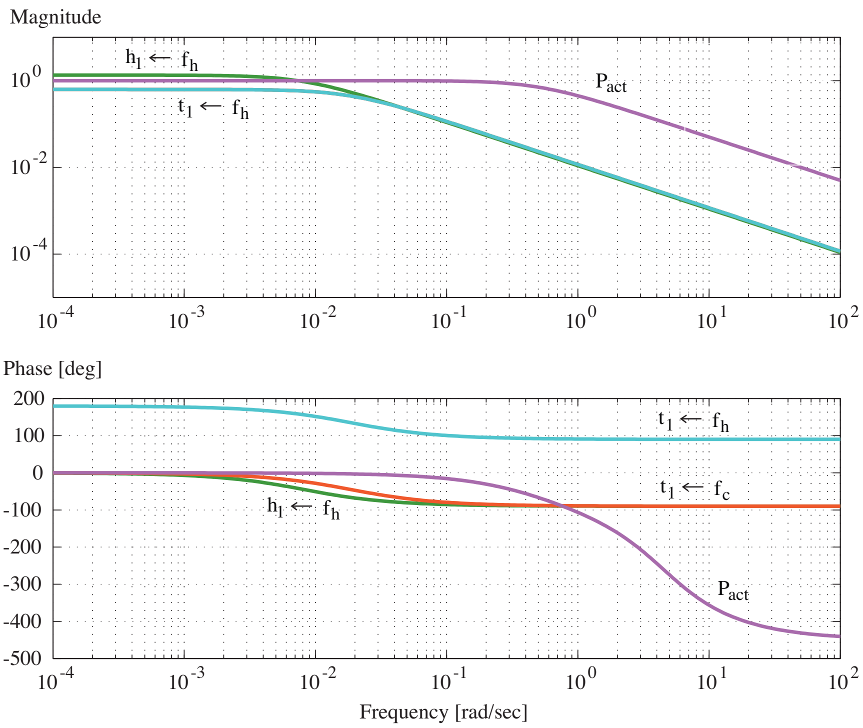
Linearized Tank 1 dynamics

$$G_{\text{nom}}(s) = \left[\begin{array}{cc|cc} \dfrac{-(1+\beta/h_1)}{A_1\alpha} & \dfrac{\beta t_1}{A_1 h_1} & \dfrac{t_h}{A_1} & \dfrac{t_c}{A_1} \\ 0 & \dfrac{-1}{A_1\alpha} & \dfrac{1}{A_1\alpha} & \dfrac{1}{A_1\alpha} \\ \hline 0 & \alpha & 0 & 0 \\ \dfrac{1}{h_1} & \dfrac{-t_1\alpha}{h_1} & 0 & 0 \end{array}\right].$$

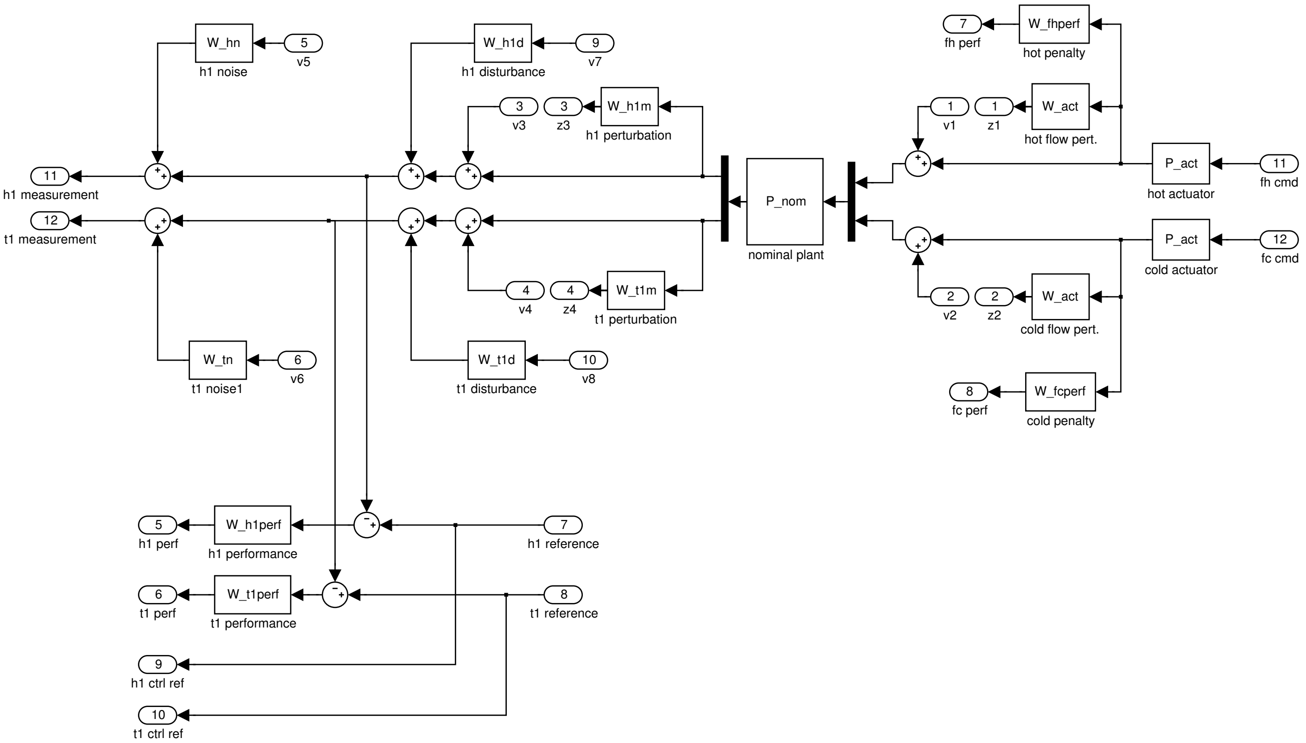
Actuator model

$$G_{\text{act}}(s) = \frac{(s^2 - 8s + 21.3)}{(2s+1)(s^2+8s+21.3)}$$ (includes a Padé delay approximation)

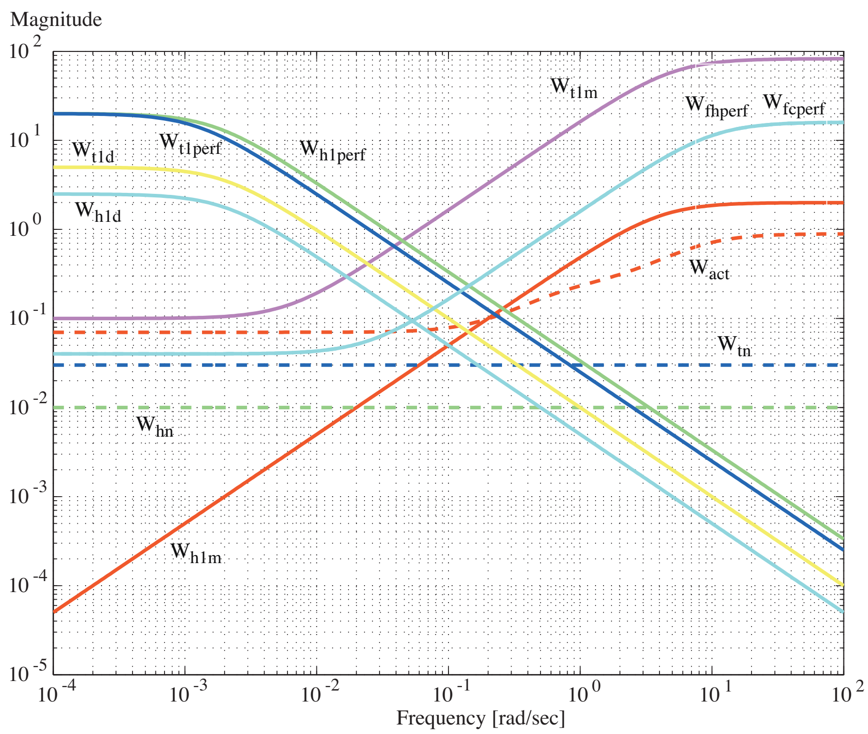
Nominal plant



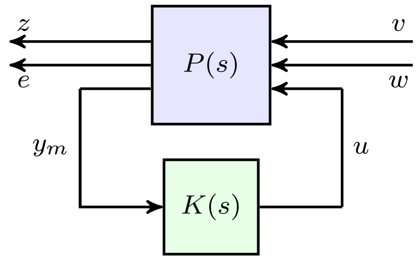
Interconnection structure: P



Weights

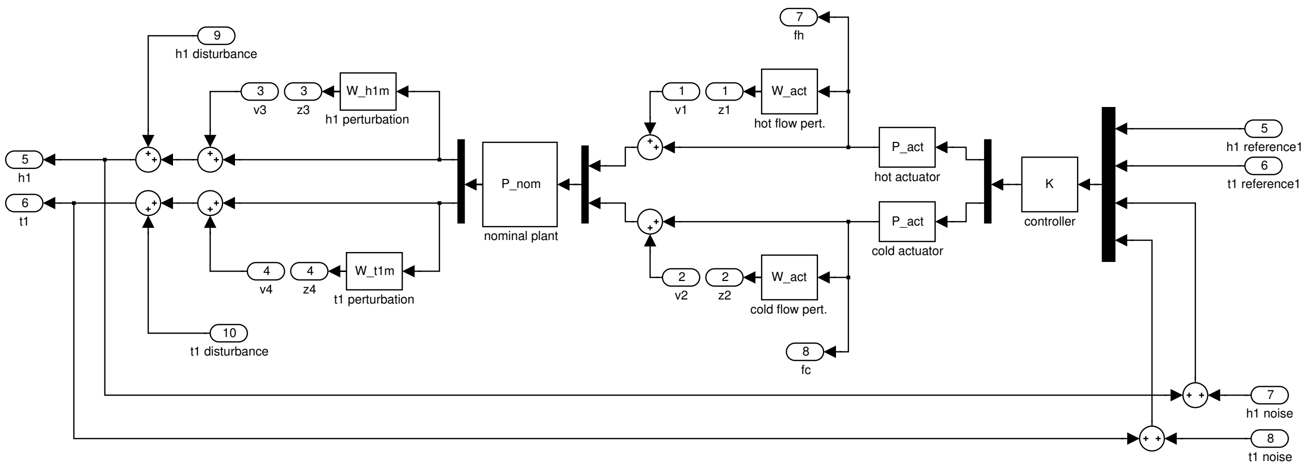


$\mathcal{H}_\infty$ controller design:



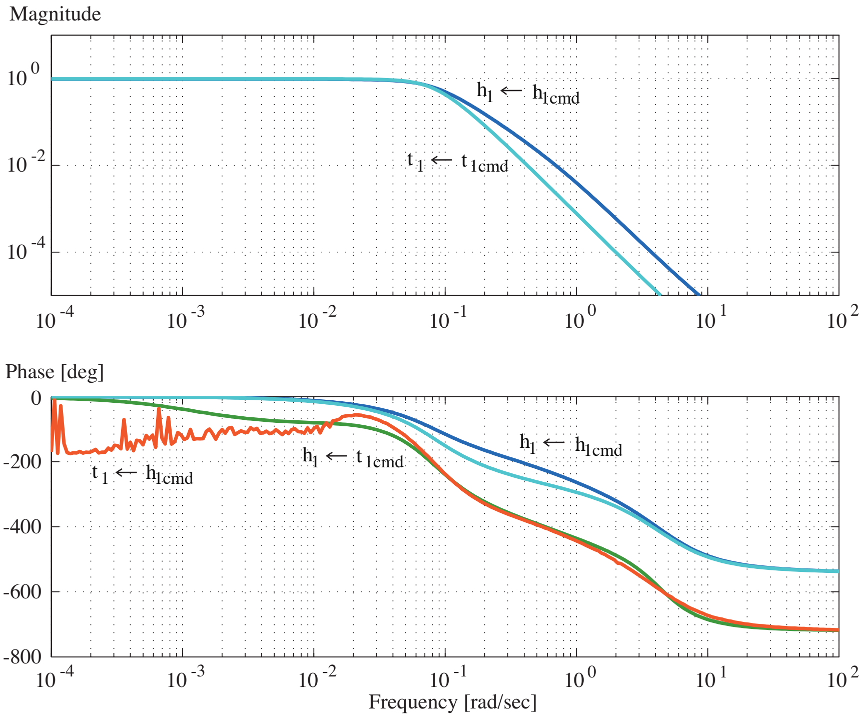

```
[A_P,B_P,C_P,D_P] = linmod('tank_model');
P = ss(A_P,B_P,C_P,D_P);
Iz = [1:4]';             % Create indices for each block.
Iv = [1:4]';
Ie = [5:8]';
Iw = [5:10]';
Iy = [9:12]';
Iu = [11:12]';
Pnomdesign = P([Ie;Iy],[Iw;Iu]);    % select [e;y] <- [w;u]
Pnomdesign = minreal(Pnomdesign);   % remove non-minimal states.

[K_inf,N_inf,gamma,info] = hinfsyn(Pnomdesign,nmeas,nctrl,...
                           'METHOD','ric',...    % Riccati solution
                           'TOLGAM',0.1);        % gamma tolerance
```


Closed-loop analysis & simulation

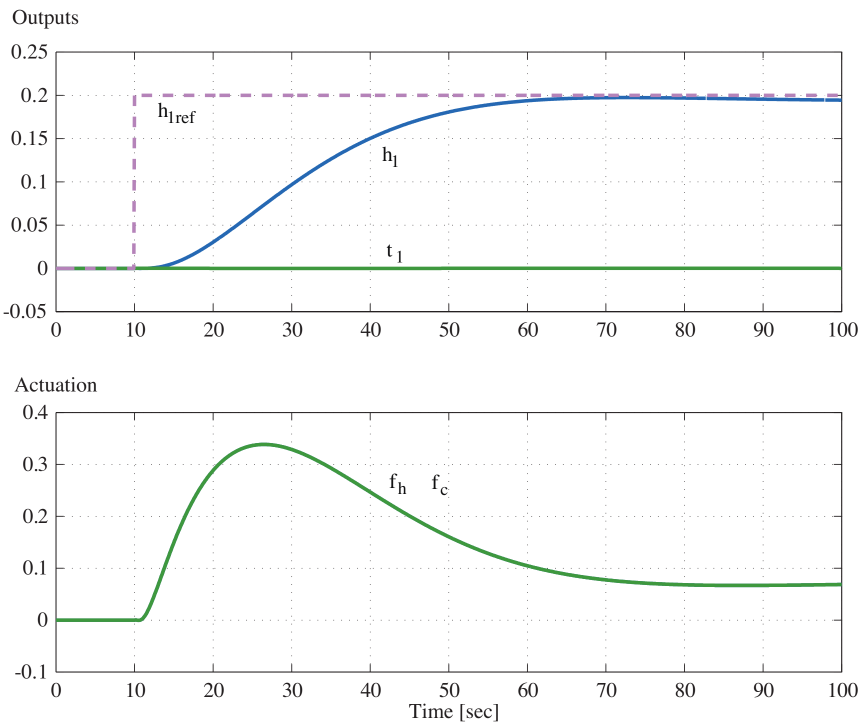



```
K = K_inf;                    % set controller for simulink block

[Aclp,Bclp,Cclp,Dclp] = linmod('tank_unweighted_clp');
N_inf = ss(Aclp,Bclp,Cclp,Dclp);

N_inf_nom = N([5:8],[5:10]);
N_inf_nom = minreal(N_inf_nom);     % remove delta weight states.
```

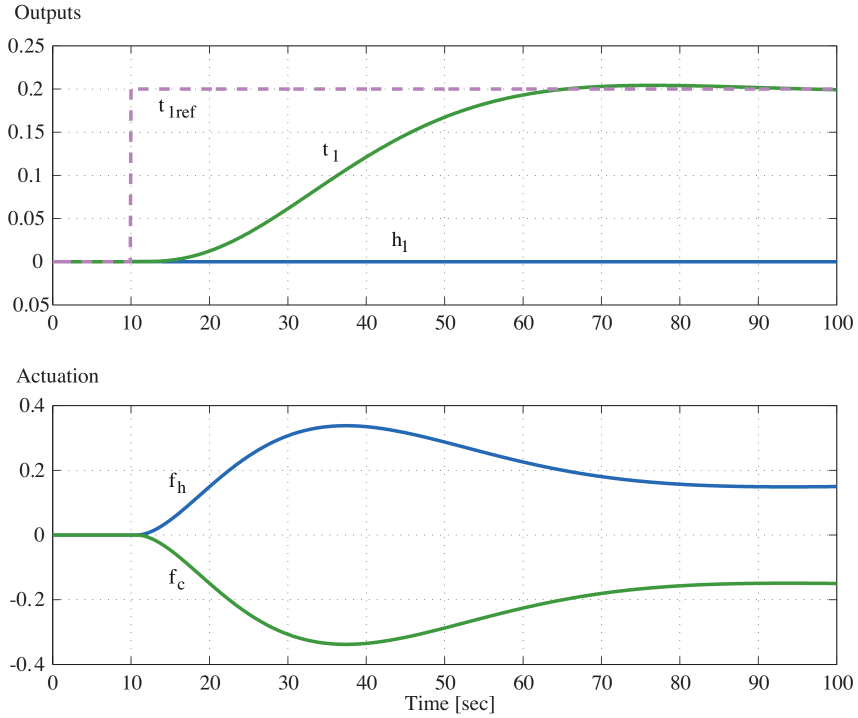
Nominal command responses



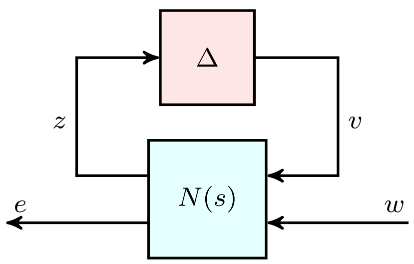
Nominal h_1 step response



Nominal t_1 step response

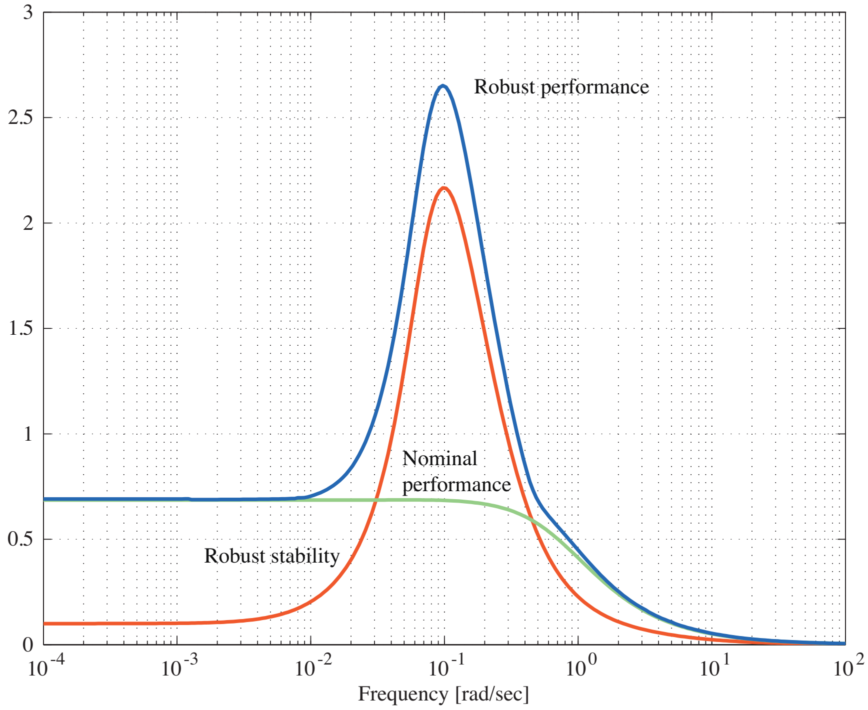


Closed-loop robustness analysis




```
N_inf = lft(P,K_inf);

omega = logspace(-4,2,250);                % frequency vector
N_inf_w = frd(N_inf,omega);                 % frequency response

muRS = mussv(N_inf_w(Iz,Iv),RS_blk);       % robust stability
muNP = svd(N_inf_w(Ie,Iw));                % nominal performance
[muRP,muinfo0] = mussv(N_inf_w,RP_blk);    % robust performance
```

$\mathcal{H}_\infty$ design — robustness analysis: $\mu_{\mathbf{\Delta}}(N(j\omega))$

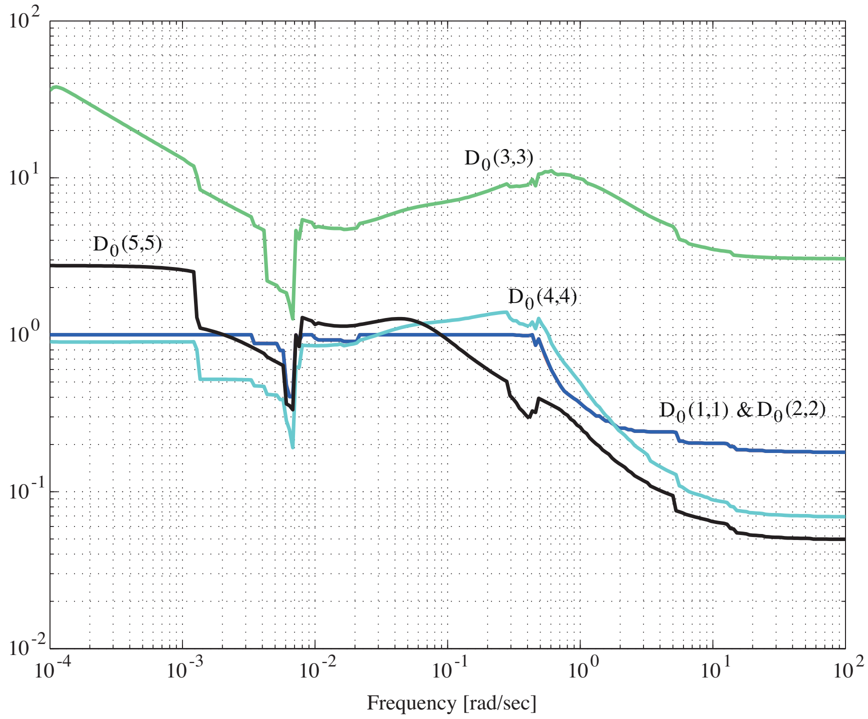


Robustness analysis: μ upper bound

$$\mu\left(\mathtt{N\_inf\_w}(j\omega)\right) \quad \leq \quad \overline{\sigma}\left(D_o(j\omega)\,\mathtt{N\_inf\_w}(j\omega)\,D_o^{-1}(j\omega)\right).$$

$$D_o(j\omega) \;=\; \begin{bmatrix} D_0(1,1) & 0 & \cdots & \cdots & 0 \\ 0 & D_0(2,2) & & & \vdots \\ \vdots & & D_0(3,3) & & \vdots \\ \vdots & & & D_0(4,4) & 0 \\ 0 & \cdots & \cdots & 0 & D_0(5,5)I_4 \end{bmatrix}.$$

μ upper bound: D scalings ("raw")



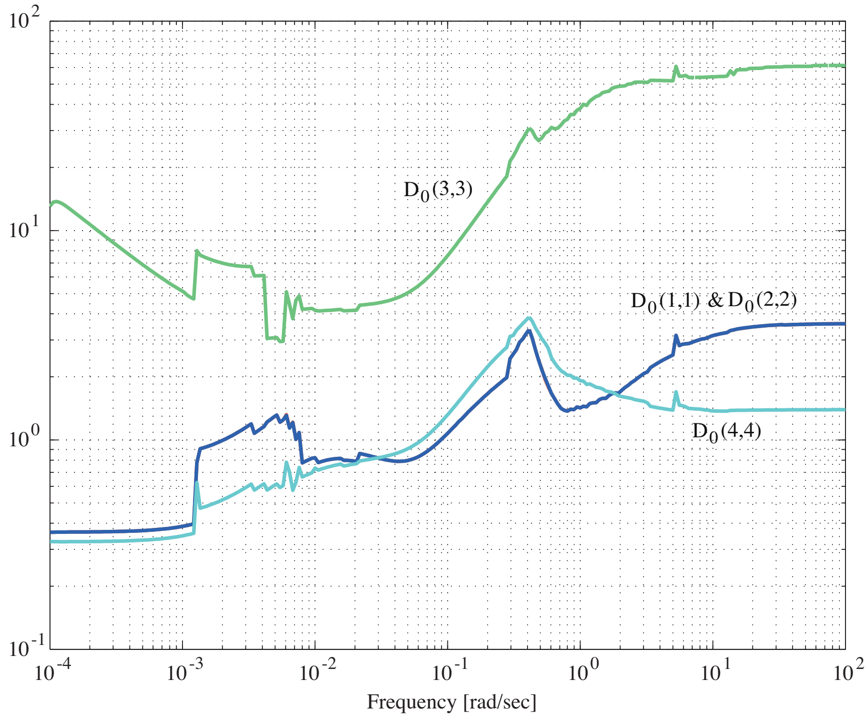
Robustness analysis: normalised μ upper bound

$$\mu\left(\texttt{N\_inf\_w}(j\omega)\right) \quad \leq \quad \overline{\sigma}\left(D_o(j\omega)\,\texttt{N\_inf\_w}(j\omega)\,D_o^{-1}(j\omega)\right)$$

$$D_o(j\omega) = \begin{bmatrix} D_0(1,1)/D_0(5,5) & 0 & \cdots & \cdots & 0 \\ 0 & D_0(2,2)/D_0(5,5) & & & \vdots \\ \vdots & & D_0(3,3)/D_0(5,5) & & \vdots \\ \vdots & & & D_0(4,4)/D_0(5,5) & 0 \\ 0 & \cdots & \cdots & 0 & I_4 \end{bmatrix}.$$

```
[Dl0,Dr0] = mussvunwrap(muinfo0);      % extract D-scales

D0_perf = Dl0(5,5);
D0_1 = Dl0(1,1)/D0_perf;      % normalize w.r.t. perf. D-scale
D0_2 = Dl0(2,2)/D0_perf;
D0_3 = Dl0(3,3)/D0_perf;
D0_4 = Dl0(4,4)/D0_perf;
```

μ upper bound: D scalings normalised $(D_0([5:8],[5:8]) = I)$

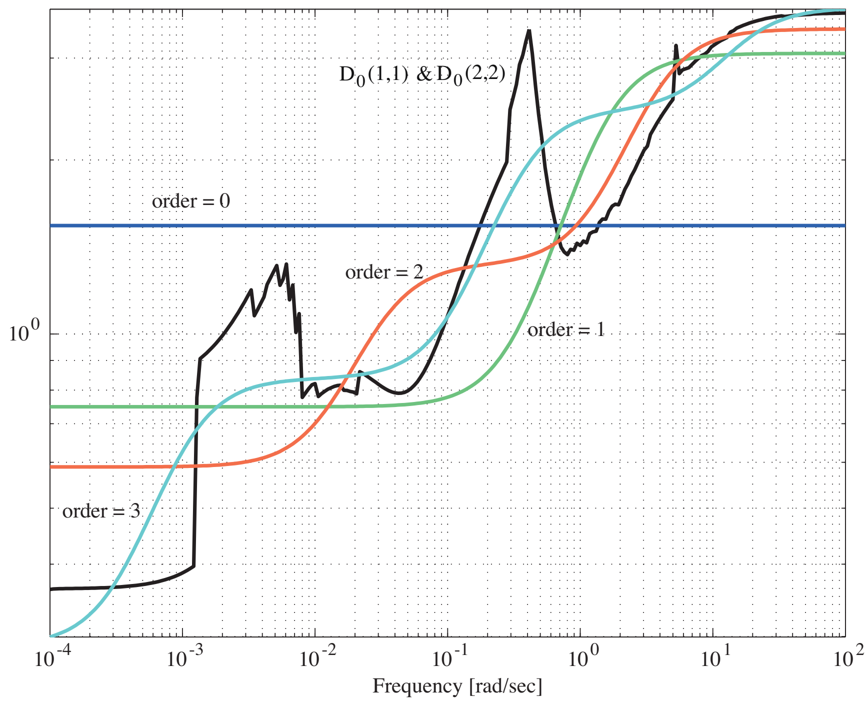


μ upper bound: Fitting D-scales: $\hat{D}_0(s)$

$$\underbrace{\hat{D}_0(i,i)(s)\mid_{s=j\omega}}_{\text{frequency response}} \approx \underbrace{D_0(i,i)(j\omega)}_{\text{calculated } D\text{-scale}}, \qquad i = 1,\ldots,4.$$

```
DO_1a = fitfrd(genphase(DO_1),0);        % 0th order fit
DO_1b = fitfrd(genphase(DO_1),1);        % 1st order fit
DO_1c = fitfrd(genphase(DO_1),2);        % 2nd order fit
DO_1d = fitfrd(genphase(DO_1),3);        % 3rd order fit
```

μ iteration #1: Fitting: $\hat{D}_0(1,1)(s)\big|_{s=j\omega} \approx D_0(1,1)(j\omega)$



μ iteration #1 accuracy: $D_0(1,1)$ fit

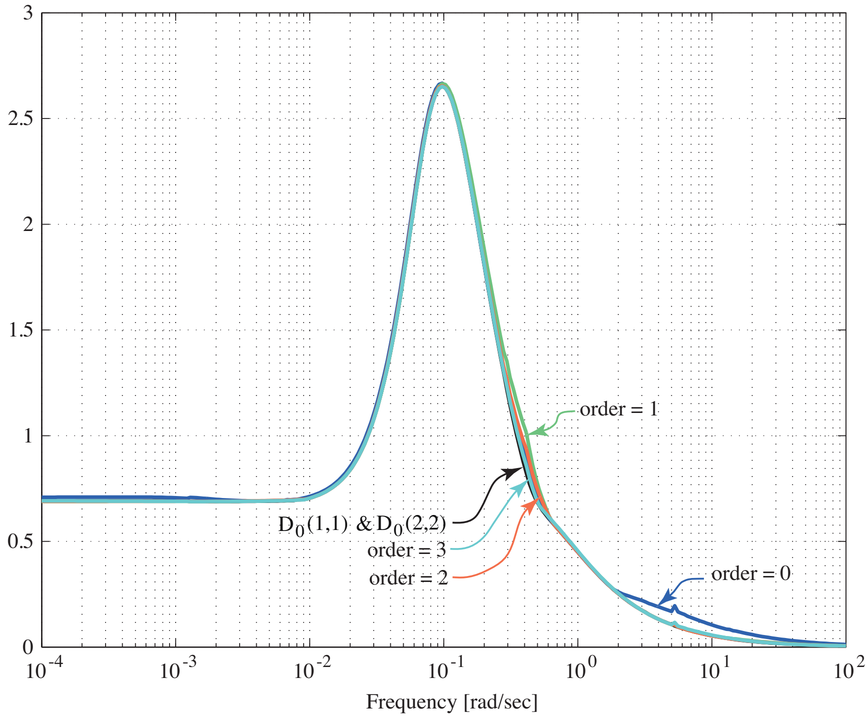
Check $\hat{D}_0(1,1)(s)\big|_{s=j\omega} \approx D_0(1,1)(j\omega)$ w.r.t. max. singular value

$$\overline{\sigma}\left(\begin{bmatrix} \hat{D}_0(1,1) & & & 0 \\ & D_0(2,2) & & \\ & & \ddots & \\ 0 & & & I_4 \end{bmatrix} \text{N\_inf\_w}(j\omega) \begin{bmatrix} \hat{D}_0^{-1}(1,1) & & & 0 \\ & D_0^{-1}(2,2) & & \\ & & \ddots & \\ 0 & & & I_4 \end{bmatrix}\right)$$

$$\overset{?}{\approx}$$

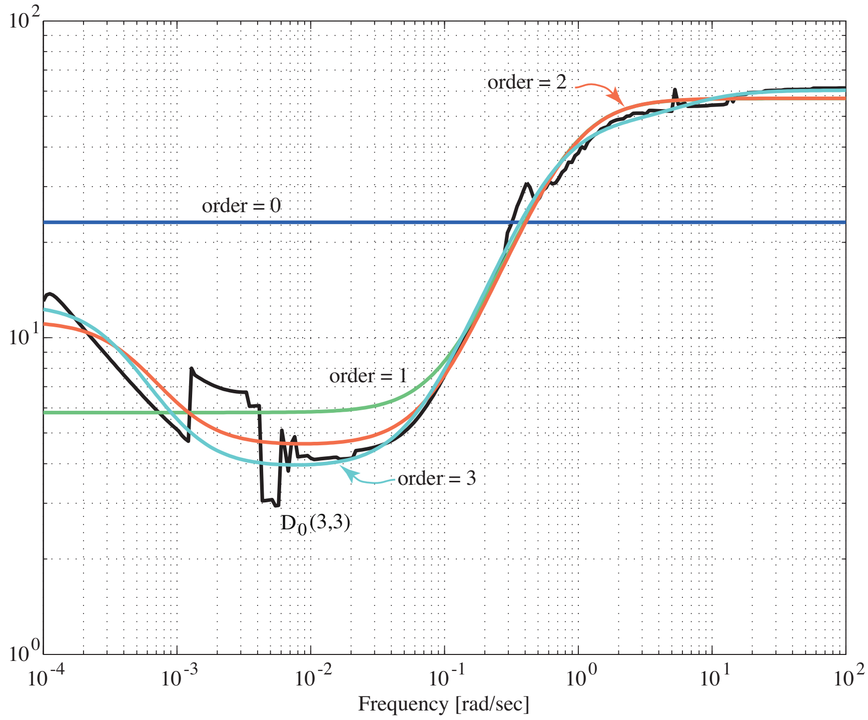
$$\overline{\sigma}\left(\begin{bmatrix} D_0(1,1) & & & 0 \\ & D_0(2,2) & & \\ & & \ddots & \\ 0 & & & I_4 \end{bmatrix} \text{N\_inf\_w}(j\omega) \begin{bmatrix} D_0^{-1}(1,1) & & & 0 \\ & D_0^{-1}(2,2) & & \\ & & \ddots & \\ 0 & & & I_4 \end{bmatrix}\right)$$

μ iteration #1 accuracy: $D_0(1,1)$ fit

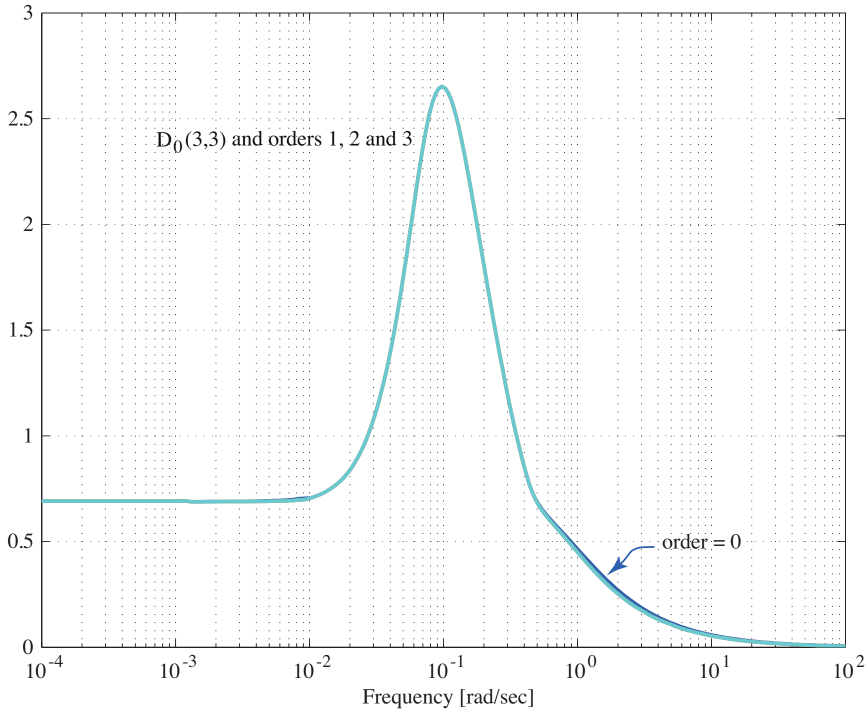


Select 0^{th} order

μ iteration #1: Fitting: $\hat{D}_0(3,3)(s)\big|_{s=j\omega} \approx D_0(3,3)(j\omega)$

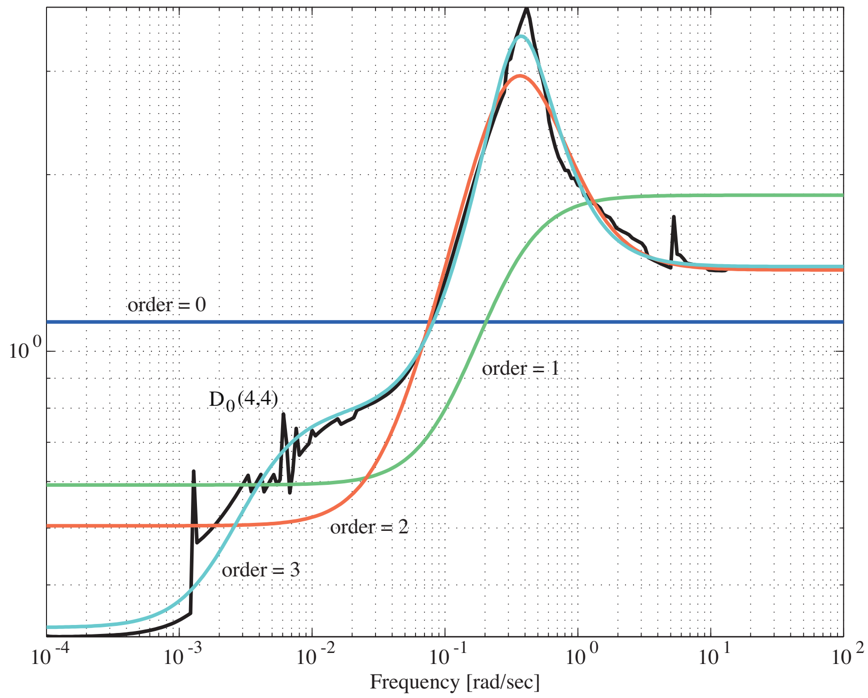


μ iteration #1 accuracy: $D_0(3,3)$ fit

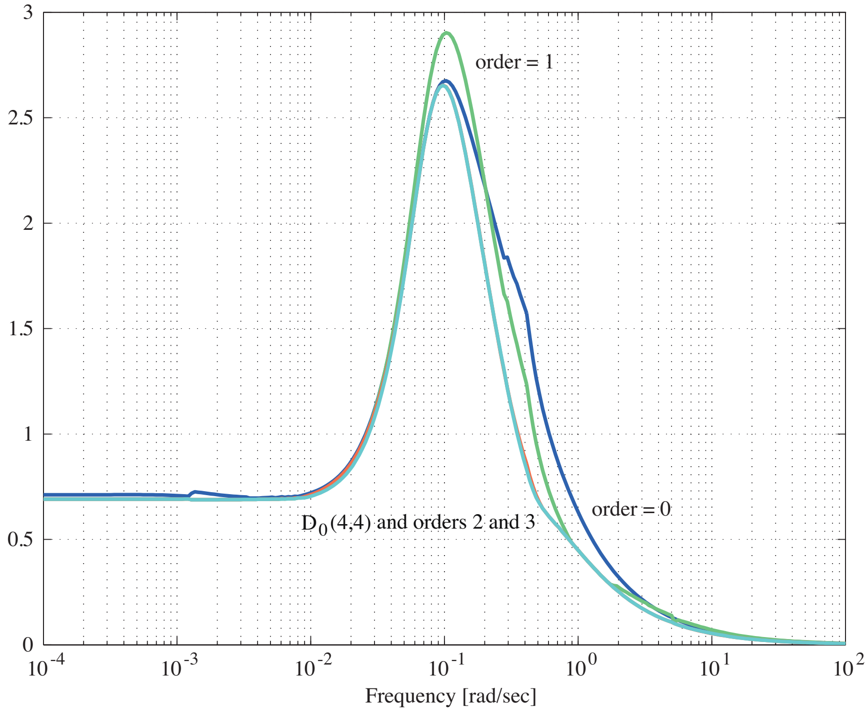


Select 0^{th} order

μ iteration #1: Fitting: $\hat{D}_0(4,4)(s)\big|_{s=j\omega} \approx D_0(4,4)(j\omega)$

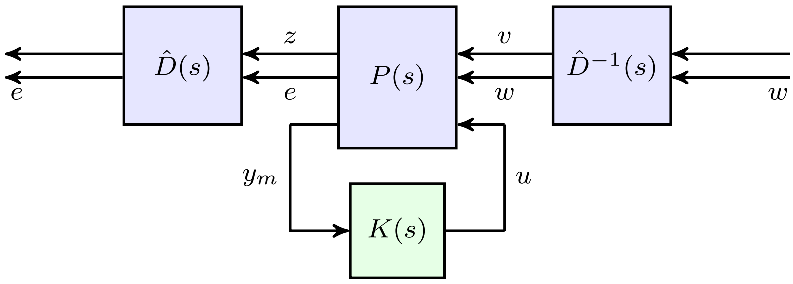


μ iteration #1 accuracy: $D_0(4,4)$ fit



Select 2^{nd} order

μ design iteration #1: Redesign controller for scaled plant

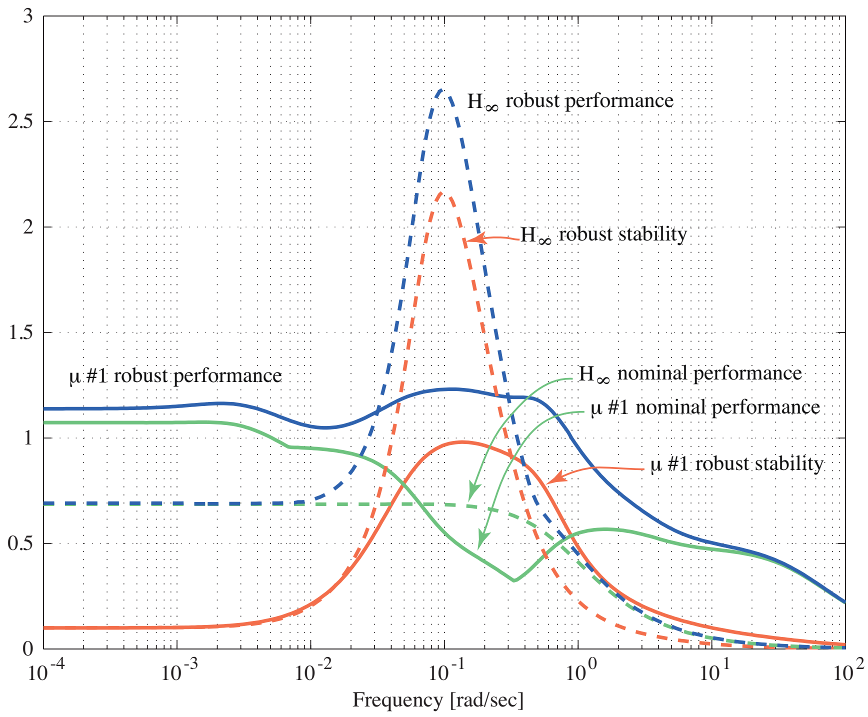



```
Pmu1design = [sysD0, zeros(nz,ne+nmeas);
               zeros(ne,nz), eye(ne,ne), zeros(ne,nmeas);
               zeros(nmeas,nz+ne),eye(nmeas,nmeas)] ...
                        * P * ...
              [sysDi0,zeros(nv,nw+nctrl);
               zeros(nw,nv), eye(nw,nw), zeros(nw,nctrl);
               zeros(nctrl,nv+nw),eye(nctrl,nctrl)];

[Kmu1,Nmu1,gamma1,info1] = hinfsyn(Pmu1design,nmeas,nctrl,...
                           'METHOD','lmi',...     % LMI solution
                           'TOLGAM',0.1);         % gamma tolerance
```

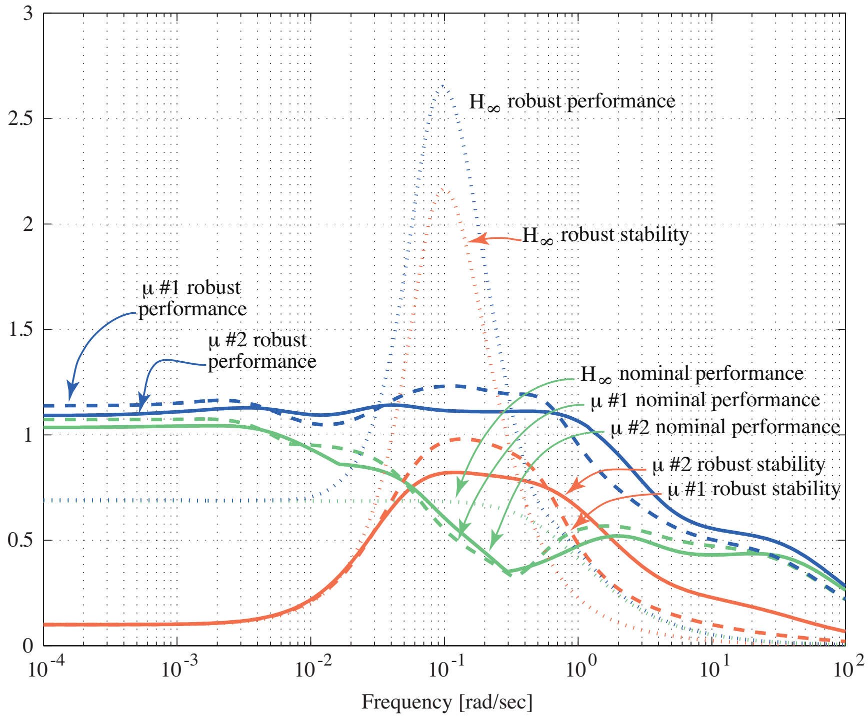
μ design iteration #1: Robustness analysis for Kmu1

```
Nmu1 = lft(P,Kmu1);              % repeat the robustness analysis
Nmu1_w = frd(Nmu1,omega);

muRS1 = mussv(Nmu1_w(Iz,Iv),RS_blk);
muNP1 = svd(Nmu1_w(Ie,Iw));
[muRP1,muinfo1] = mussv(Nmu1_w,RP_blk);
```

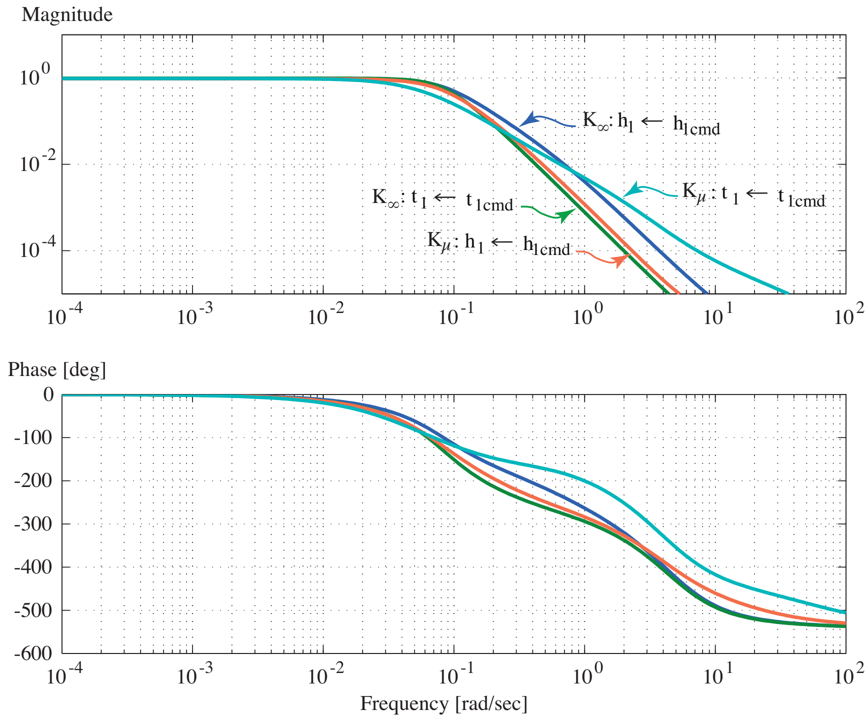

μ design iteration #1: Robustness analysis



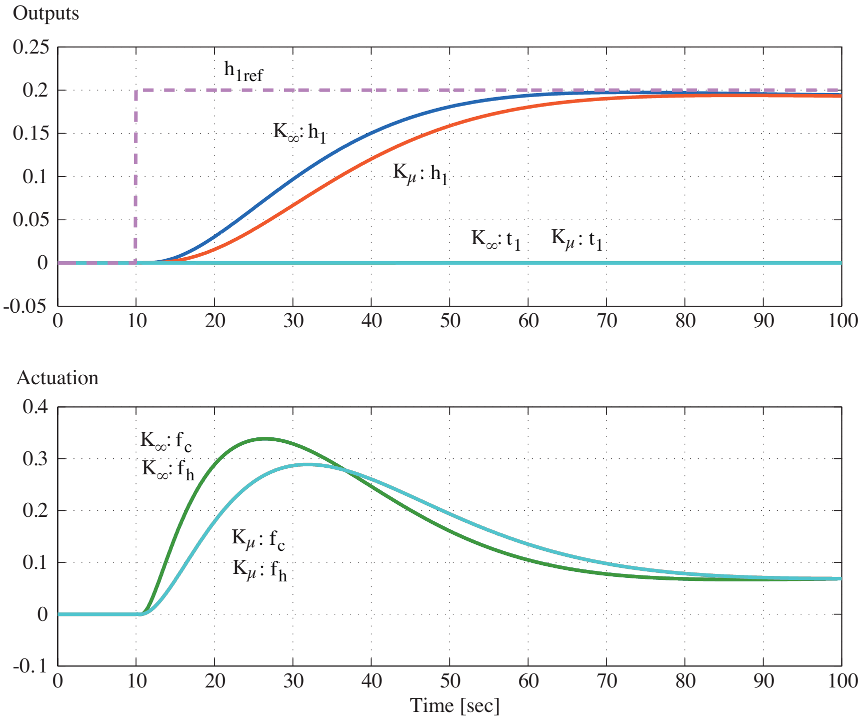
μ design iteration #2: robustness analysis



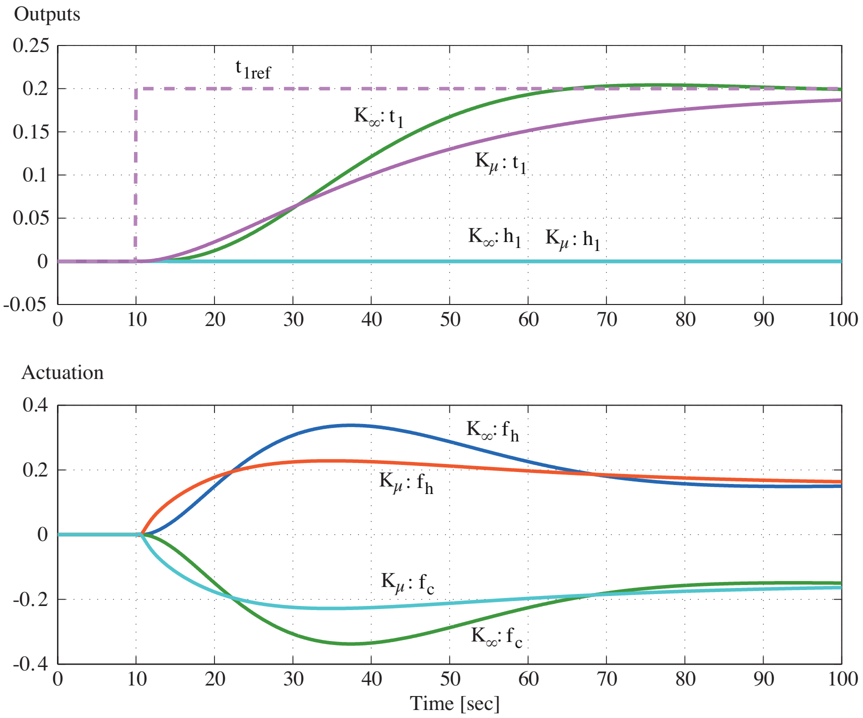
$\mathcal{H}_\infty$ and μ controllers: nominal command responses



$\mathcal{H}_\infty$ and μ controllers: nominal h_1 step responses



$\mathcal{H}_\infty$ and μ controllers: nominal t_1 step responses



Perturbed plant: extracting a “worst-case” perturbation

```
% Find the peak of the mu plot for the initial controller.

mudata = frdata(muRP);                  % extract data
maxmu = max(mudata);                    % find the peak
maxidx = find(maxmu == max(maxmu));     % find index for max over omega
maxidx = maxidx(1);                     % ensure only one frequency chosen.

Delta0 = mussvunwrap(muinfo0);          % Delta from Khinf analysis
Delta0data = frdata(Delta0);
Delta0data_w = Delta0data(:,:,maxidx);
```


Perturbed plant: extracting a “worst-case” perturbation

$$\Delta_0(j0.097) \quad =$$

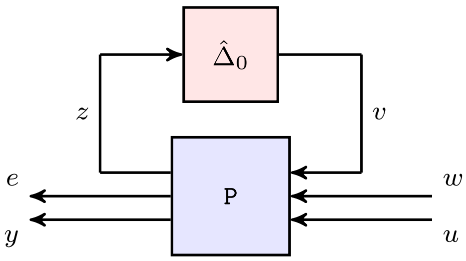
$$\begin{bmatrix}
0.08-0.37j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.08-0.37j & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.34+0.17j & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.37-0.06j & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.00-0.00j & -0.00+0.00j & -0.00-0.00j & 0.00+0.00j \\
0 & 0 & 0 & 0 & -0.00-0.00j & -0.01-0.01j & -0.01+0.02j & 0.01-0.02j \\
0 & 0 & 0 & 0 & -0.00+0.00j & -0.00+0.00j & 0.00+0.00j & -0.00-0.00j \\
0 & 0 & 0 & 0 & 0.00+0.00j & 0.01+0.14j & 0.23-0.03j & -0.23+0.03j \\
0 & 0 & 0 & 0 & 0.00+0.00j & 0.00+0.00j & 0.00-0.00j & -0.00+0.00j \\
0 & 0 & 0 & 0 & 0.00-0.00j & 0.02-0.03j & -0.06-0.03j & 0.06+0.03j
\end{bmatrix}$$

Perturbed plant: fit a transfer function to the perturbation

```
Delta0_wc = ss(zeros(nv,nz));
for i = 1:4,
  delta_i = Delta0data_w(i,i);
  gamma = abs(delta_i);
  if imag(delta_i) > 0,
    delta_i = -1*delta_i;
    gamma = -1*gamma;
  end
  x = real(delta_i)/abs(gamma);                  % fit a Pade with
  tau = 2*omega(maxidx)*(sqrt((1+x)/(1-x)));     % the same phase
  Delta0_wc(i,i) = gamma*(-s + tau/2)/(s+tau/2);
end

nDelta = norm(Delta0data_w);     % the size should be 1/mu.
Delta0_wc = Delta0_wc/nDelta;    % scale perturbation to size = 1.
```

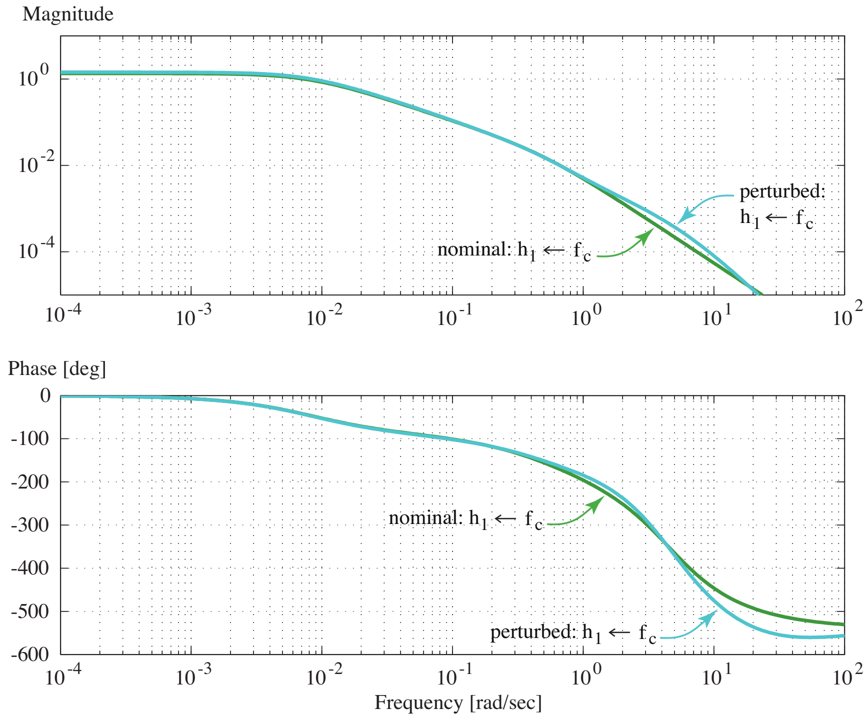

Perturbed plant: close the loop around Δ

$$\hat{\Delta}_0(s) = \left[\begin{array}{cccc|cccc} -0.12 & 0 & 0 & 0 & 0.6629 & 0 & 0 & 0 \\ 0 & -0.12 & 0 & 0 & 0 & 0.6629 & 0 & 0 \\ 0 & 0 & -0.02332 & 0 & 0 & 0 & 0.3314 & 0 \\ 0 & 0 & 0 & -0.0080 & 0 & 0 & 0 & 0.1657 \\ \hline 0.3621 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0.3621 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -0.1407 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.0969 & 0 & 0 & 0 & -1 \end{array}\right]$$

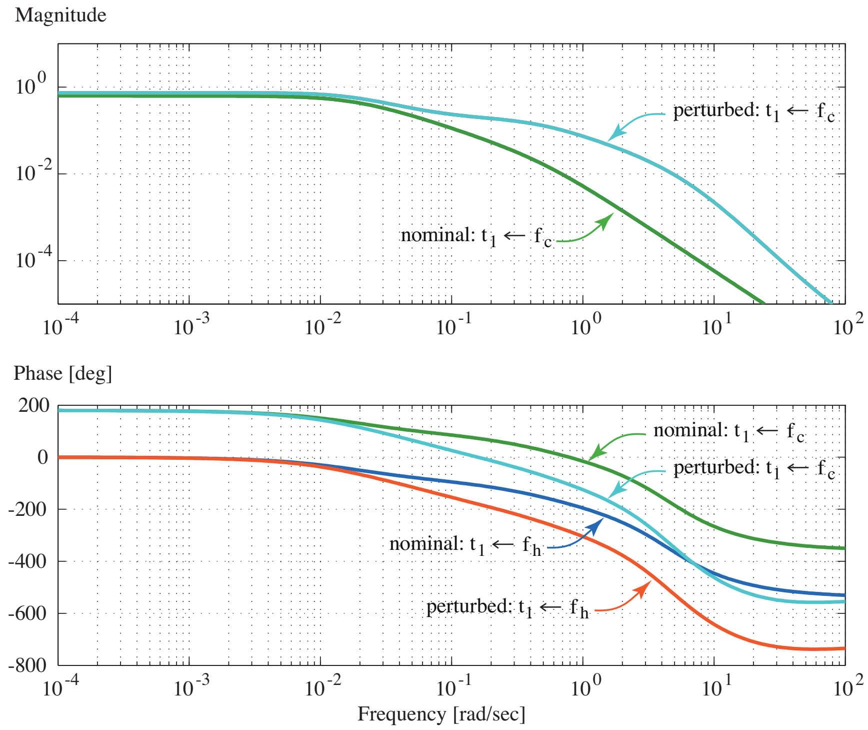



```
Ppert = lft(Delta0_wc,P([Iz;Iy],[Iv;Iu]));     % perturbed y <- u
Ppert_f = frd(Ppert,omega);
```

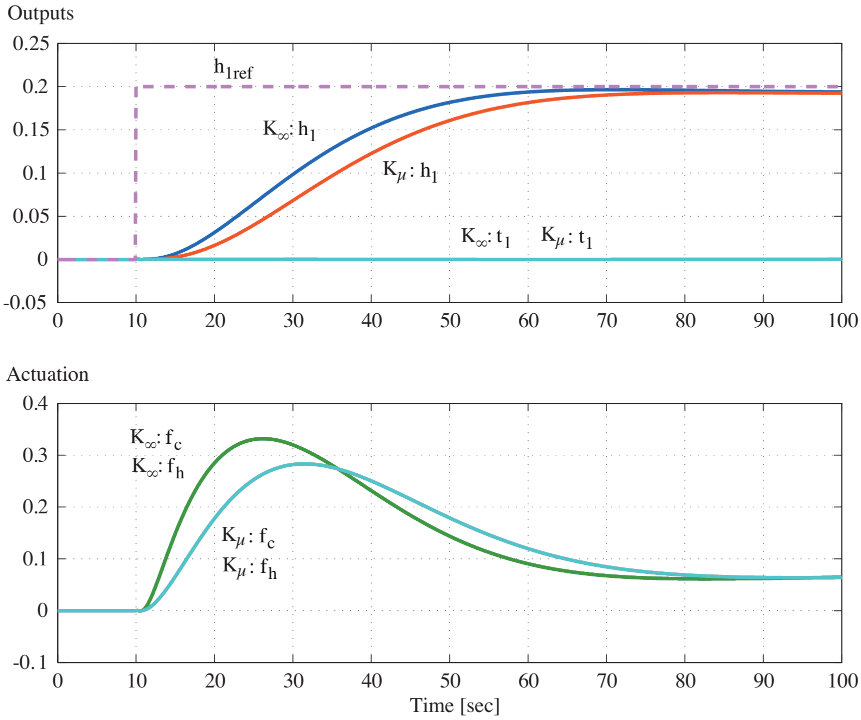
Perturbed plant: nominal and perturbed h_1 responses



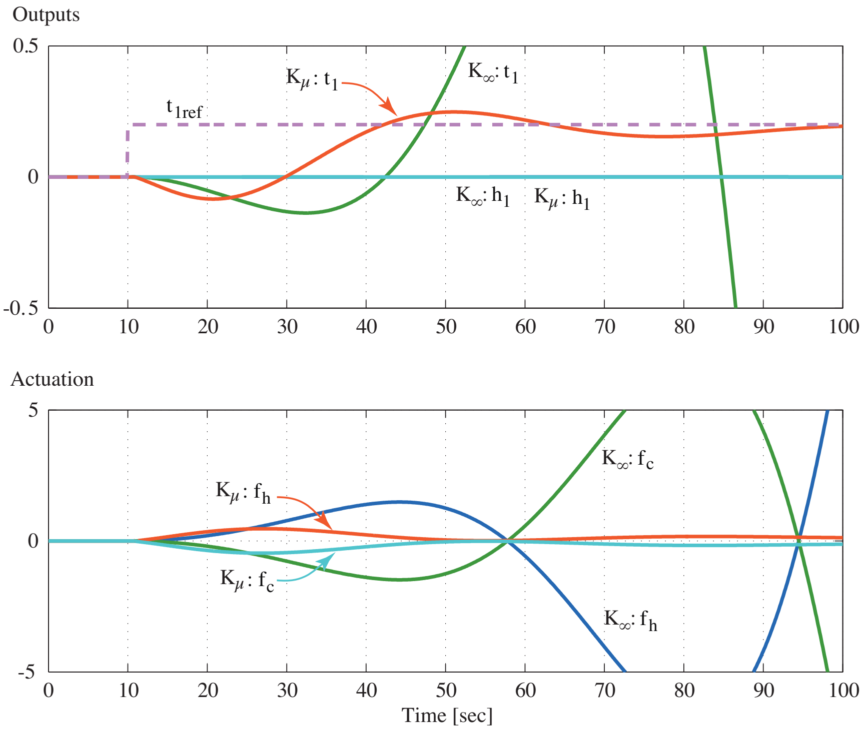
Perturbed plant: nominal and perturbed t_1 responses



Perturbed plant: nominal and perturbed h_1 step responses



Perturbed plant: nominal and perturbed t_1 step responses



Notes and references

Skogestad & Postlethwaite (2nd Ed.)

General control formulation: section 3.8

System norms: section 4.10

LQG design: section 9.2

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis: section 9.3

D-K iteration: section 8.12

MATLAB

`hinfsyn, h2syn` Robust Control Toolbox documentation