Control Systems 2

Lecture 10: Robust stability and performance for MIMO systems

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8:15, Wednesday 4th May, 2022

Robustness analysis

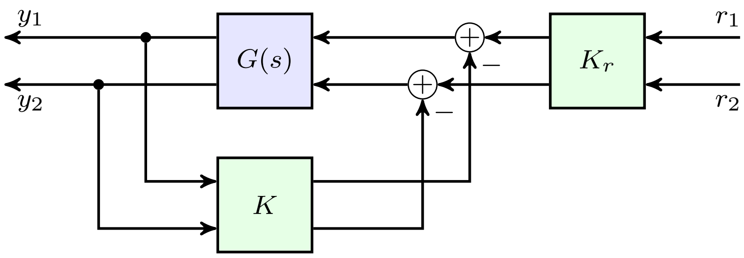
Nominal stability (NS) Is the closed-loop system stable when the plant is known exactly?

Robust stability (RS) Is the closed-loop system stable when there is uncertainty in our knowledge of the plant?

Nominal performance (NP) Does the closed-loop system meet the performance specifications when the plant is known exactly?

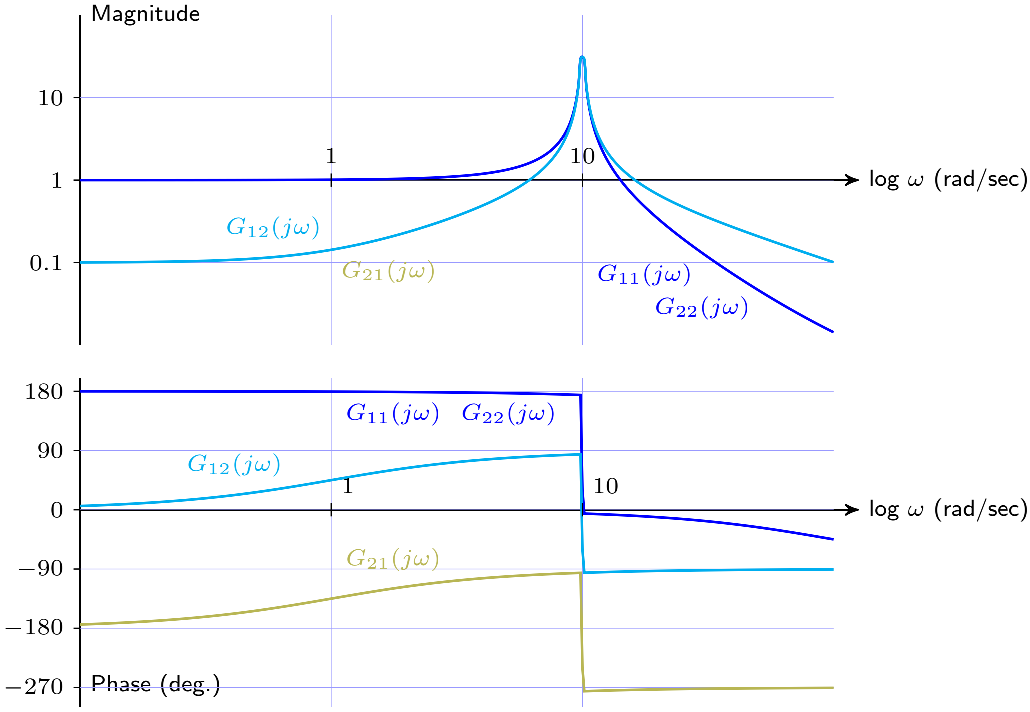
Robust performance (RP) Does the closed-loop system meet the performance specifications when there is uncertainty in our knowledge of the plant?

Robustness analysis example: rotating satellite



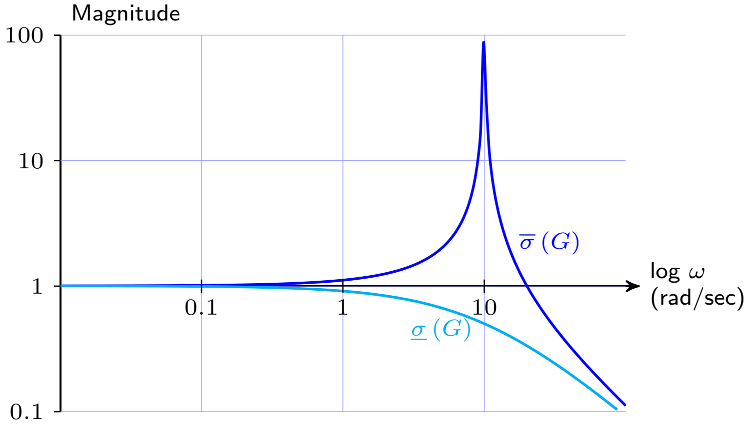
$$G(s) = \frac{1}{s^2 + \alpha^2} \begin{bmatrix} s - \alpha^2 & \alpha(s+1) \\ -\alpha(s+1) & s - \alpha^2 \end{bmatrix}$$

Spinning satellite: Bode plots

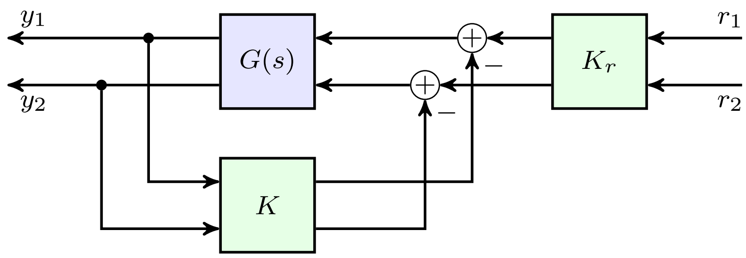


Singular values: spinning satellite

$$G(s) = \frac{1}{s^2+\alpha^2}\begin{bmatrix} s-\alpha^2 & \alpha(s+1) \\ -\alpha(s+1) & s-\alpha^2 \end{bmatrix}, \qquad \alpha = 10.$$



An example: rotating satellite

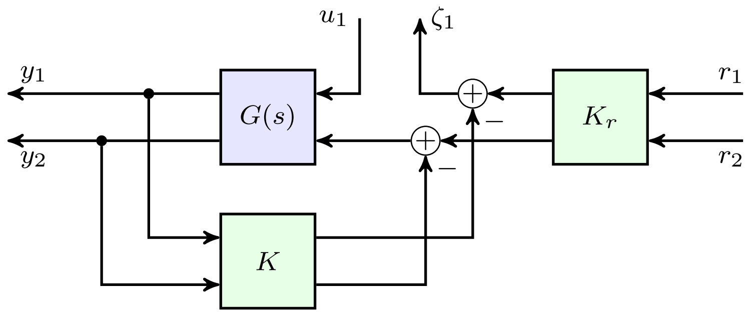


$$K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad K_r = \frac{1}{1+\alpha^2}\begin{bmatrix} 1 & -\alpha \\ \alpha & 1 \end{bmatrix}$$ ($\alpha = 10$ in plots)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = (I + G(s)K)^{-1}\, G(s)K_r \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$= \frac{1}{s+1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

Loop-at-a-time analysis



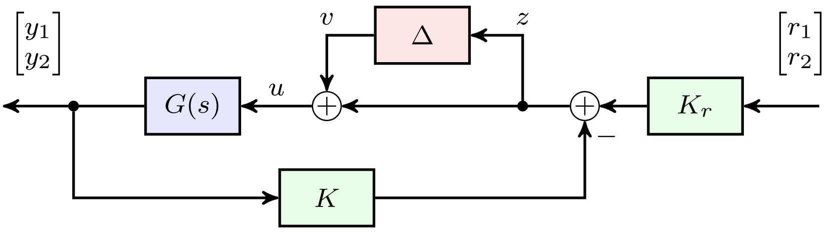
Loop transfer function:

$$\zeta_1 = \frac{-1}{s} u_1$$

Margins:

Gain margin:	∞
Phase margin:	90 degrees

Robustness analysis

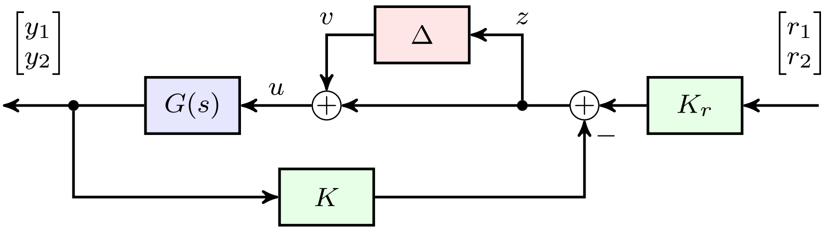


Perturbations (actuator uncertainty):

$$u_1 = (1 + \delta_1) z_1$$
$$u_2 = (1 + \delta_2) z_2$$

$$\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$$

Robustness analysis

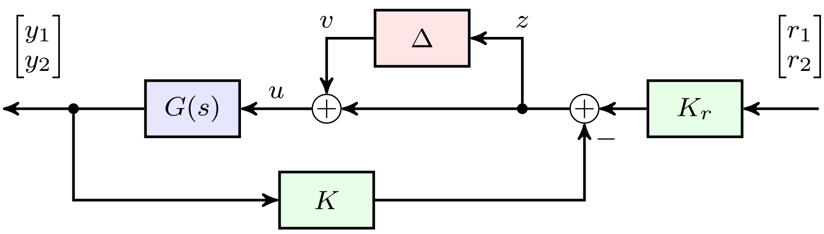


Closed-loop characteristic polynomial:

$$s^2 + (2 + \delta_1 + \delta_2)s + [1 + \delta_1 + \delta_2 + (\alpha^2 + 1)\delta_1\delta_2] = 0$$

If $\delta_2 = 0$ the smallest destabilizing δ_1 is $\delta_1 = -1$.

Robustness analysis



Closed-loop characteristic polynomial:

$$s^2 + (2 + \delta_1 + \delta_2)s + [1 + \delta_1 + \delta_2 + (\alpha^2 + 1)\delta_1\delta_2] = 0$$

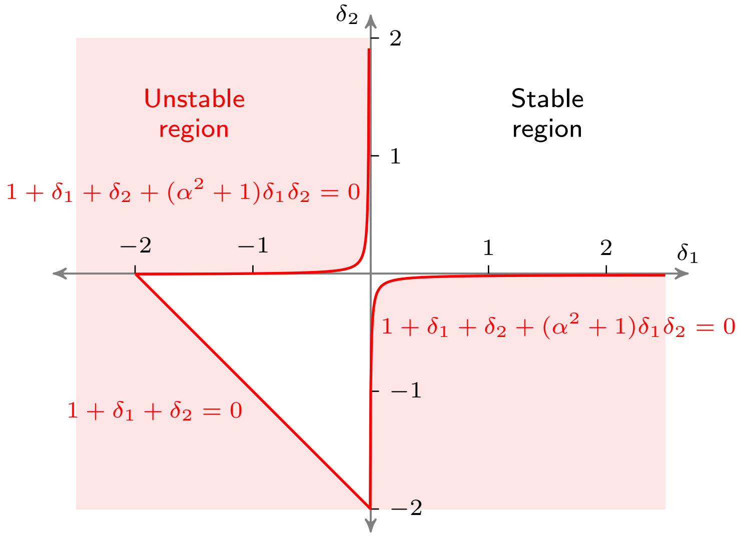
If $\delta_2 = 0$ the smallest destabilizing δ_1 is $\delta_1 = -1$.

But it can be destabilized by a much smaller choice of δ_1, δ_2:

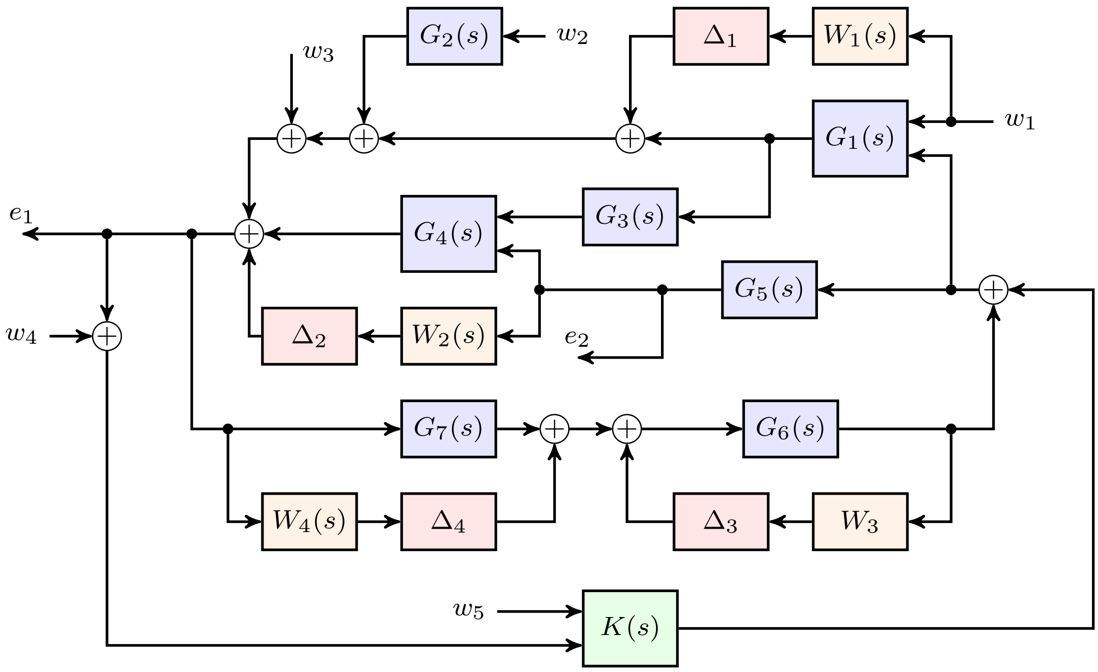
$$\delta_1 = \frac{1}{\sqrt{\alpha^2 + 1}} \approx 0.1 \quad \text{and} \quad \delta_2 = -\delta_1.$$

Robustness analysis

Stability region

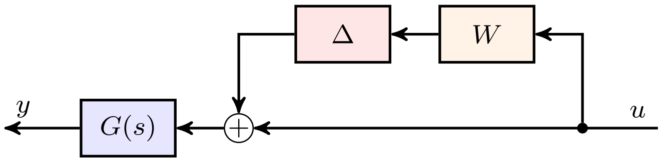


Systems with multiple perturbations

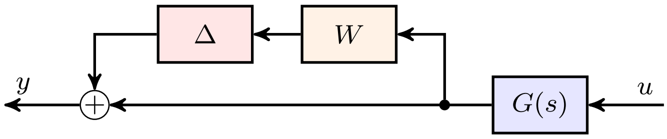


Common perturbation structures

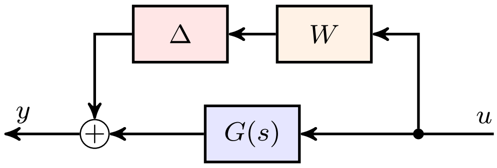
Input perturbation:



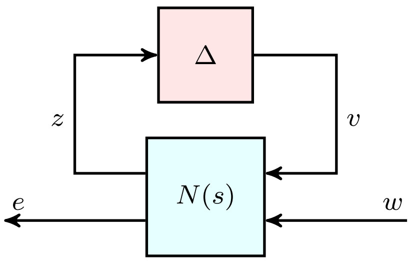
Output perturbation:



Additive perturbation:



General structure for robustness analysis

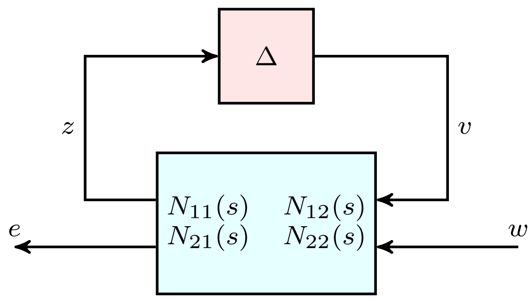


$N(s)$ is a *stable* closed-loop interconnection

Δ is a block-diagonal matrix of perturbations.

$$N(s) = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix} \qquad \Delta = \begin{bmatrix} \Delta_1 & & 0 \\ & \ddots & \\ 0 & & \Delta_m \end{bmatrix}$$

General structure for robustness analysis



Robust case:

$$\begin{aligned} e &= \left[N_{21}(s)\Delta\left(I - N_{11}(s)\Delta\right)^{-1} N_{12}(s) + N_{22}(s)\right]\, w \\ &= \mathcal{F}_u\left(N(s), \Delta\right)\, w \end{aligned}$$

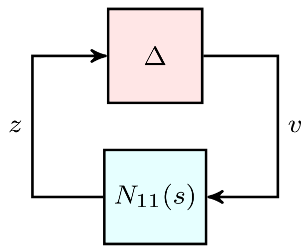
Nominal case: if $\Delta = 0$ then $e = N_{22}(s)\, w$.

Robust stability analysis

Δ is an unknown stable system.

$\boldsymbol{\Delta}$ defines the "structure" of Δ.

$\Delta \in \boldsymbol{\Delta}$



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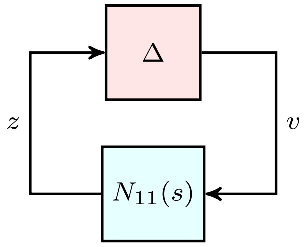

Is the interconnection stable for given Δ?

Δ is destabilizing $\iff$ $\det(I - N_{11}(j\omega)\Delta(j\omega))$ encircles 0.

Robust stability analysis

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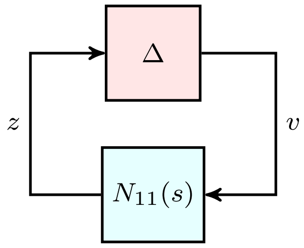


Is the interconnection stable for given Δ?

Δ is destabilizing $\iff$ $\det(I - N_{11}(j\omega)\Delta(j\omega))$ encircles 0.

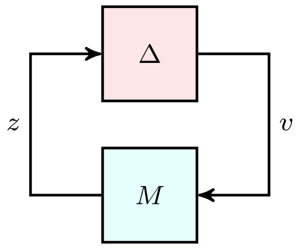
What is the smallest destabilising Δ?

$\min_{\Delta \in \boldsymbol{\Delta}} \overline{\sigma}(\Delta)$ such that $\det(I - N_{11}(j\omega))\Delta(j\omega)) = 0$

Structured singular value

Δ is a complex matrix.

M is a complex matrix.

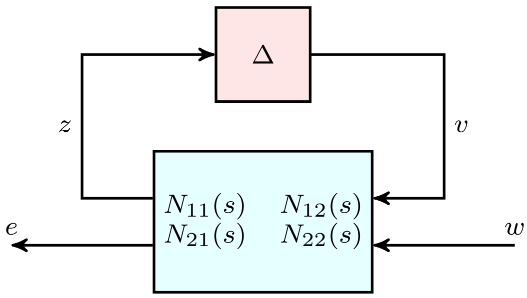


Definition:

$$\mu_{\boldsymbol{\Delta}}(M) = \begin{cases} 0 \text{ if no } \Delta \in \boldsymbol{\Delta} \text{ solves } \det(I - M\Delta) = 0 \\ \text{otherwise} \\ \left[\min_{\Delta \in \boldsymbol{\Delta}} \left\{ \beta \;\middle|\; \exists \Delta, \|\Delta\| \le \beta, \det(I - M\Delta) = 0 \right\} \right]^{-1} \end{cases}$$

$= 1/$size of smallest destabilizing Δ.

Robust stability



The closed-loop interconnection is stable for all
$\Delta \in \boldsymbol{\Delta}$, $\|\Delta\|_{\mathcal{H}_\infty} \le 1$,

if and only if

$$\mu_{\boldsymbol{\Delta}}(N_{11}(j\omega)) < 1 \quad \text{for all } \omega.$$

Properties of μ

If an invertible matrix D commutes with all $\Delta \in \mathbf{\Delta}$, then,

$$\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(DMD^{-1})$$

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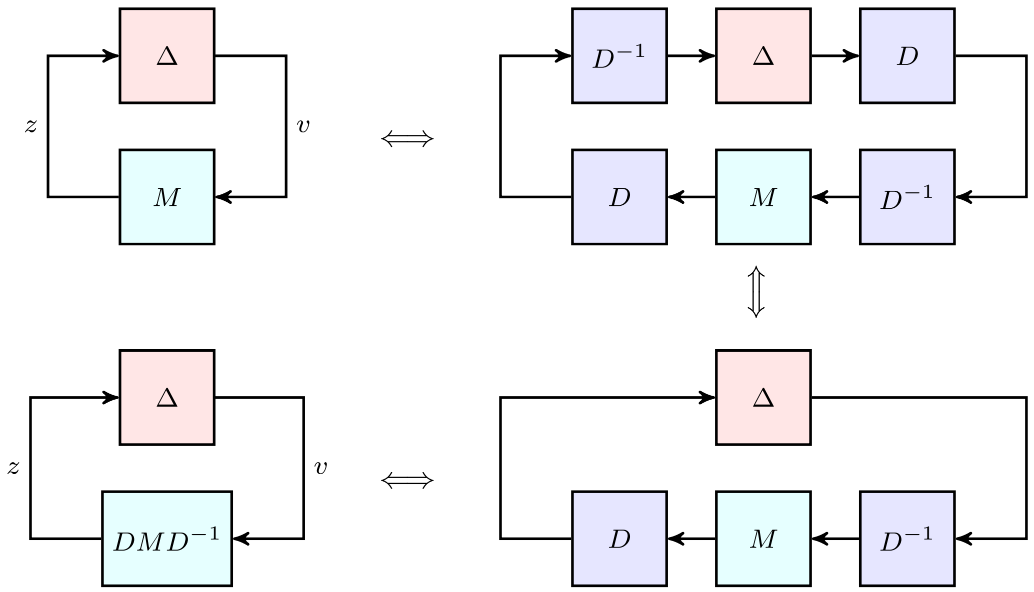


Properties of μ

For all possible structures, $\mathbf{\Delta}$,

$$\mu_{\mathbf{\Delta}}(M) \;\leq\; \overline{\sigma}(M) \qquad \text{"unstructured case"}$$

So, for every invertible matrix commuting with $\mathbf{\Delta}$,

$$\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(DMD^{-1}) \leq \overline{\sigma}\left(DMD^{-1}\right)$$

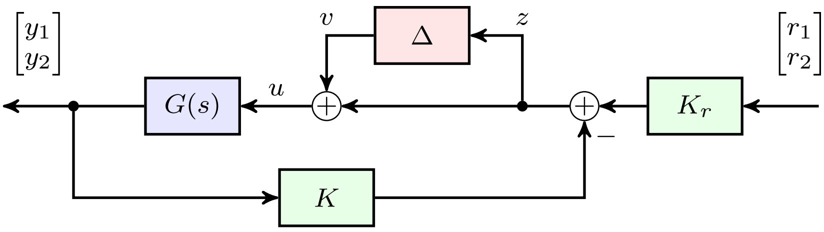
An upper bound:

Search over all invertible matrices that commute with all $\Delta \in \mathbf{\Delta}$.
Notation: $\mathcal{D} = \{\, D \mid D\Delta D^{-1} \in \mathbf{\Delta} \text{ for all } \Delta \in \mathbf{\Delta} \,\}$

$$\mu_{\mathbf{\Delta}}(M) \leq \inf_{D \in \mathcal{D}} \overline{\sigma}\left(DMD^{-1}\right)$$

See `mussv` command in MATLAB.

Satellite example (revisited)



The transfer function for v to z can be seen via,

$$z = (I + KG(s))^{-1}K_r\, r - (I + KG(s))^{-1}KG(s)\, v,$$

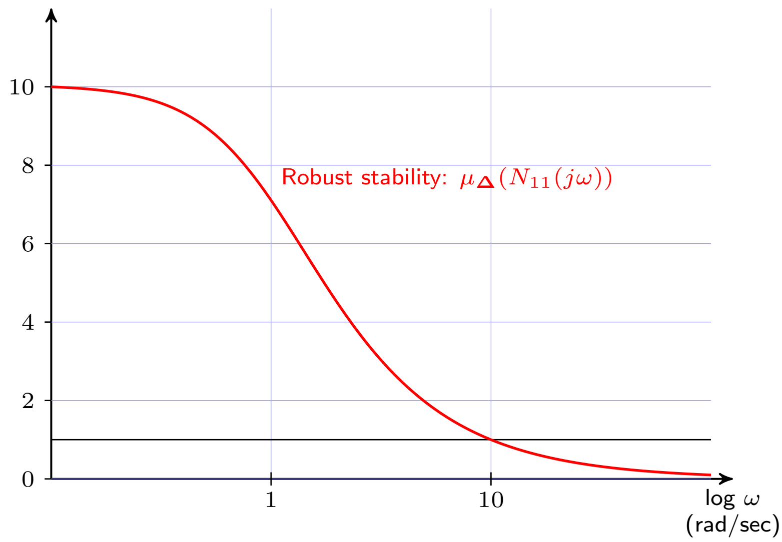
so,

$$N_{11}(s) = -(I + KG(s))^{-1}KG(s).$$

Robust stability (for perturbations up to size 1)

$$\text{RS} \quad \Longleftrightarrow \quad \mu_{\mathbf{\Delta}}\left(N_{11}(j\omega)\right) < 1 \text{ for all } \omega, \qquad \mathbf{\Delta} = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$$

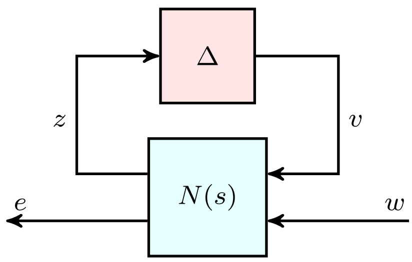
Robust stability: satellite example



Because $\mu_{\mathbf{\Delta}}(N_{11}(j\omega))$ is not less than 1 for all ω,

the closed-loop system is NOT stable for all perturbations (up to size 1).
It is stable for all diagonal perturbations up to size 1/10.

Robust performance



Nominal performance ($\Delta = 0$):

$\|e\|_2 \leq 1$ for all $\|w\|_2 \leq 1$.

$\overline{\sigma}\left(N_{22}(j\omega)\right) < 1$ for all ω $\iff$ $\|N_{22}(s)\|_{\mathcal{H}_\infty} < 1$

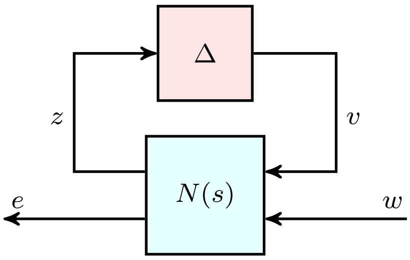
Robust performance:

$\|e\|_2 \leq 1$ for all $\|w\|_2 \leq 1$ and all $\Delta \in \mathbf{\Delta}$, $\|\Delta\|_{\mathcal{H}_\infty} < 1$.

Robust performance

As an equivalent robust stability problem

Robust performance



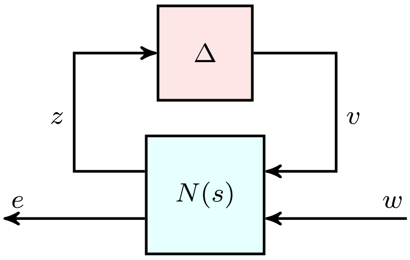
$RS:$ $\mu_{\boldsymbol{\Delta}}(N_{11}(j\omega)) < 1,$ for all ω

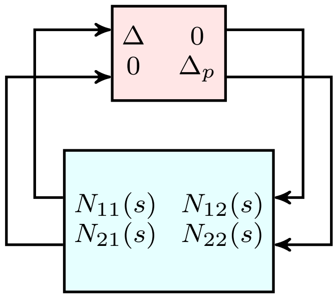
$RP:$ $\|\mathcal{F}_u\left(N(s), \Delta\right)\|_{\mathcal{H}_\infty} \leq 1$
for all $\Delta \in \boldsymbol{\Delta}$

Robust performance

As an equivalent robust stability problem

Robust performance $\iff$ Robust stability





$RS:$ $\mu_{\boldsymbol{\Delta}}(N_{11}(j\omega)) < 1,$ for all ω

$RP:$ $\|\mathcal{F}_u\left(N(s), \Delta\right)\|_{\mathcal{H}_\infty} \leq 1$
for all $\Delta \in \boldsymbol{\Delta}$

$\mu_{\tilde{\boldsymbol{\Delta}}}(N(j\omega)) < 1,$ for all ω

$$\tilde{\boldsymbol{\Delta}} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_p \end{bmatrix}$$

$\Delta \in \boldsymbol{\Delta}, \Delta_p \in \mathcal{C}^{\dim(w)\times\dim(e)}$

Robust performance

Robust performance:

$$\|e\|_2 \leq 1 \quad \text{for all} \quad \|w\|_2 \leq 1 \text{ and all } \Delta \in \mathbf{\Delta}, \|\Delta\|_{\mathcal{H}_\infty} < 1.$$

Define a larger block structure:

$$\tilde{\mathbf{\Delta}} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_p \end{bmatrix}, \quad \Delta \in \mathbf{\Delta},\ \Delta_p \in \mathcal{C}^{\dim(w)\times\dim(e)}$$

Robust performance is satisfied if and only if,

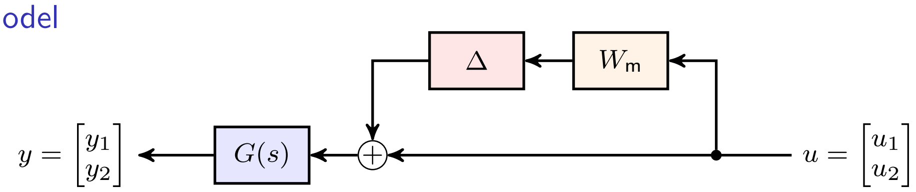
$$\mu_{\tilde{\mathbf{\Delta}}}(N(j\omega)) < 1 \quad \text{for all } \omega.$$

Robust performance example: DV distillation column (revisited)

Nominal plant

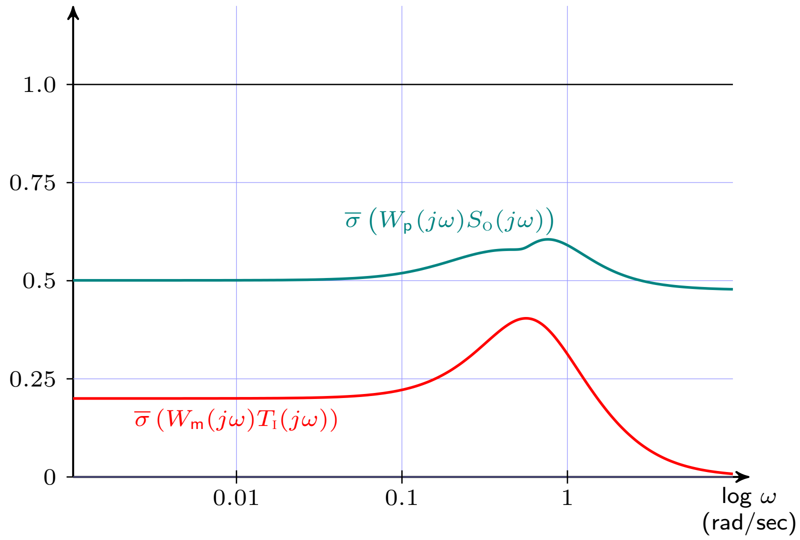
$$G(s) = \frac{1}{(100s+1)(s+1)} \begin{bmatrix} -87.8 & 1.4 \\ -108.2 & -1.4 \end{bmatrix}$$

Perturbation model



$$W_{\mathrm{m}}(s) = \frac{(s+0.2)}{(0.5s+1)} I_{2\times 2}, \qquad \Delta = \underbrace{\begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}}_{\text{"unstructured" perturbation}} \quad \text{or} \quad \Delta = \underbrace{\begin{bmatrix} \delta_{11} & 0 \\ 0 & \delta_{22} \end{bmatrix}}_{\text{"structured" perturbation}}$$

Example: nominal performance and robust stability

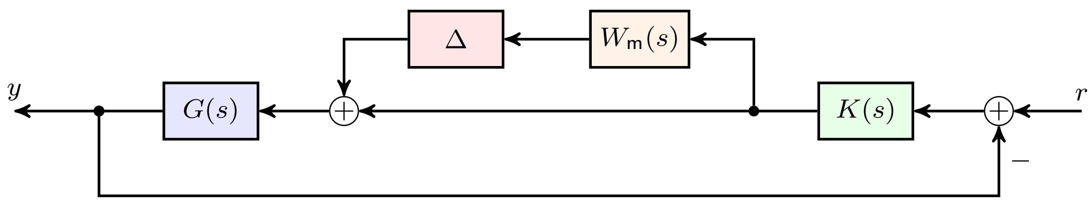


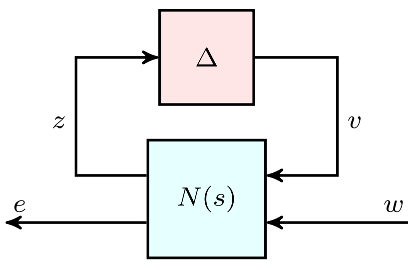
$$\boldsymbol{\Delta} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}$$

$\overline{\sigma}\left(W_{\mathsf{p}}(j\omega)S_{\mathsf{O}}(j\omega)\right) < 1,$ for all ω $\iff$ nominal performance

$\overline{\sigma}\left(W_{\mathsf{m}}(j\omega)T_{\mathsf{I}}(j\omega)\right) < 1,$ for all ω $\iff$ robust stability

Example: interconnection model





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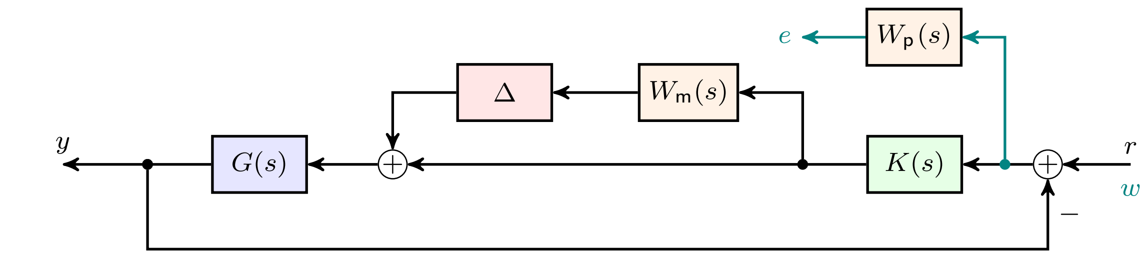

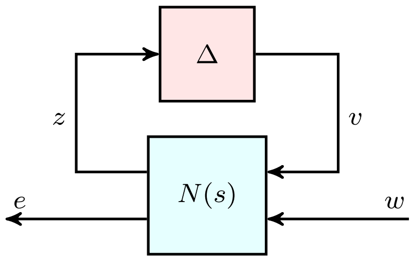

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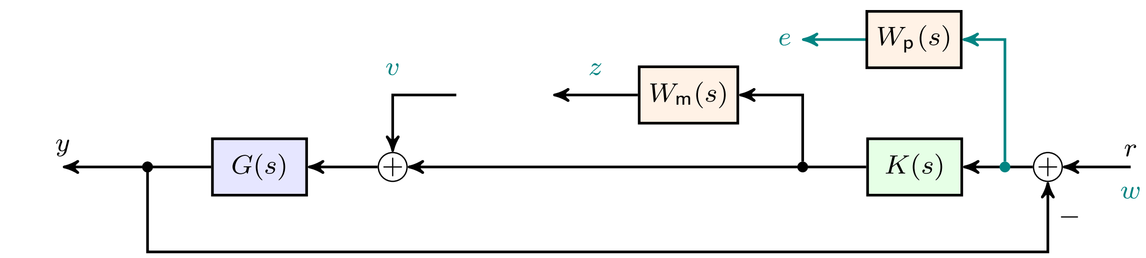
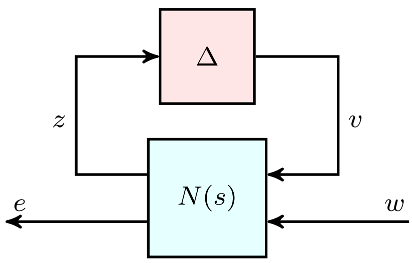
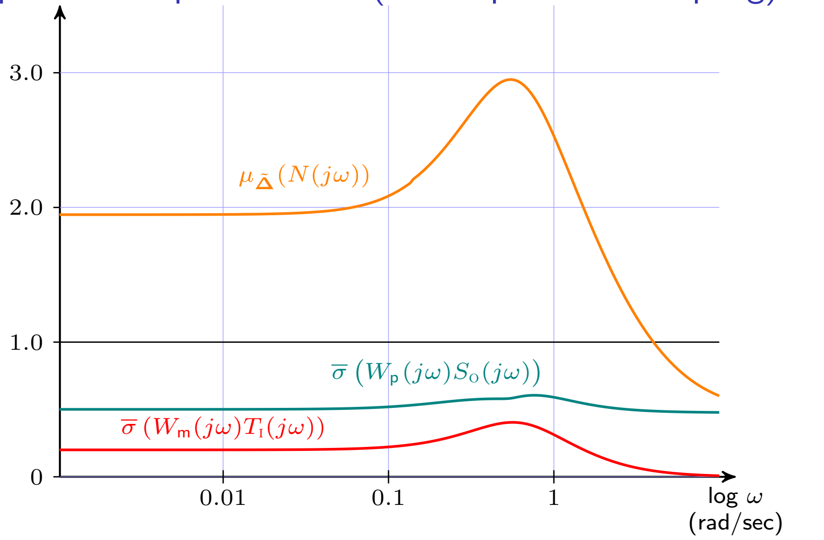




$$N_{11} = -W_m K(I + GK)^{-1} G$$
$$N_{12} = W_m K(I + GK)^{-1}$$
$$N_{21} = W_p (I + GK)^{-1} G$$
$$N_{22} = W_p (I + GK)^{-1}$$

Example: robust performance (with input cross-coupling)



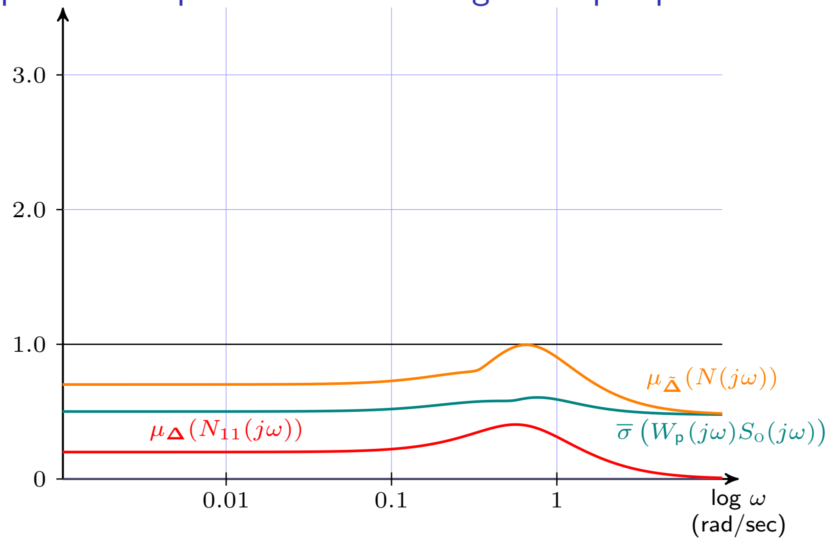
$$\boldsymbol{\Delta} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}$$

$\overline{\sigma}\left(W_{\mathsf{p}}(j\omega)S_{\mathrm{O}}(j\omega)\right) < 1,$ for all ω $\iff$ nominal performance

$\overline{\sigma}\left(W_{\mathsf{m}}(j\omega)T_{\mathrm{I}}(j\omega)\right) < 1,$ for all ω $\iff$ robust stability

$\mu_{\tilde{\boldsymbol{\Delta}}}(N(j\omega)) > 1,$ for some ω $\Longrightarrow$ robust performance violated

Example: robust performance for diagonal input perturbations



$$\boldsymbol{\Delta} = \begin{bmatrix} \delta_{11} & 0 \\ 0 & \delta_{22} \end{bmatrix}$$

$\overline{\sigma}\left(W_{\mathsf{p}}(j\omega)S_{\mathrm{O}}(j\omega)\right) < 1,$ for all ω $\iff$ nominal performance

$\mu_{\boldsymbol{\Delta}}(N_{11}(j\omega)) < 1,$ for all ω $\iff$ robust stability

$\mu_{\tilde{\boldsymbol{\Delta}}}(N(j\omega)) < 1,$ for all ω $\iff$ robust performance

Notes and references

Skogestad & Postlethwaite (2nd Ed.)

Pertubation models: sections 8.1, 8.2 and 8.3

MIMO Robust stability: sections 8.4, 8.5 and 8.6

Structured singular value ($\mu_{\boldsymbol{\Delta}}$): sections 8.7 and 8.8

Robust performance: section 8.10