Control Systems 2

Lecture 9: MIMO systems: directionality and loopshaping

Roy Smith

8:15, Wednesday 27 April, 2022

Directionality

MIMO systems: inputs and outputs are vector-valued.
Input signals are functions of both frequency and "direction."

$$u(j\omega) = \begin{bmatrix} u_1(j\omega) \\ \vdots \\ u_{n_u}(j\omega) \end{bmatrix}$$

For a given $u(s)$, the "signal gain" of the system is,

$$\frac{\|y(s)\|_2}{\|u(s)\|_2} = \frac{\|G(s)u(s)\|_2}{\|u(s)\|_2}$$

$$= \frac{\left(\int_{-\infty}^{\infty} y(j\omega)^* y(j\omega) d\omega\right)^{1/2}}{\left(\int_{-\infty}^{\infty} u(j\omega)^* u(j\omega) d\omega\right)^{1/2}} = \frac{\left(\int_{-\infty}^{\infty} \sum_{i=1}^{n_y} |y_i(j\omega)|^2 d\omega\right)^{1/2}}{\left(\int_{-\infty}^{\infty} \sum_{i=1}^{n_u} |u_i(j\omega)|^2 d\omega\right)^{1/2}}$$

Here we measure only the size of $G(s)u(s)$ (i.e. $\|G(s)u(s)\|_2$).

Singular value decomposition

Given a matrix $G \in \mathcal{C}^{m \times p}$, with $\text{rank}(G) = r \leq \min(m,p)$,

$$G = U\Sigma V^*, \quad \text{with } \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0),$$
$$\sigma_i \geq \sigma_{i+1} \geq \sigma_r > 0,$$
$$U^*U = I, \quad V^*V = I \quad \text{(unitary)}.$$

2×2 real matrix example

$$G = \underbrace{\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}}_{\substack{U \\ \text{(rotate)}}} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\substack{\Sigma \\ \text{(scale)}}} \underbrace{\begin{bmatrix} \cos\theta_2 & \pm\sin\theta_2 \\ -\sin\theta_2 & \pm\cos\theta_2 \end{bmatrix}}_{\substack{V^* \\ \text{(rotate)}}}$$

Singular value decomposition

Maximum gain; as a function of direction

$$G = U\Sigma V^* = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ 0 & & & 0 \end{bmatrix} \begin{bmatrix} v_1^* \\ \vdots \\ v_p^* \end{bmatrix}$$

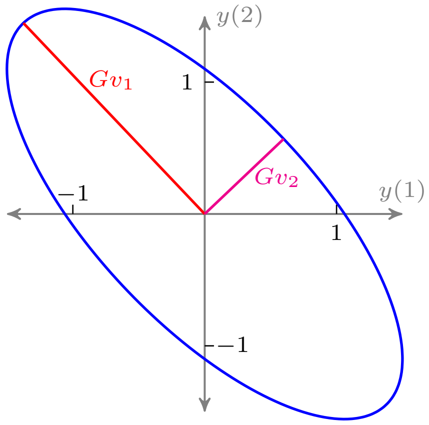
If we choose $x = v_1$ (which means $\|x\|_2 = 1$) then,

$$Gx = u_1\sigma_1, \quad \text{and } \|Gx\|_2 = \sigma_1 \; = \; \max_{\|x\|_2=1} \|Gx\|_2.$$

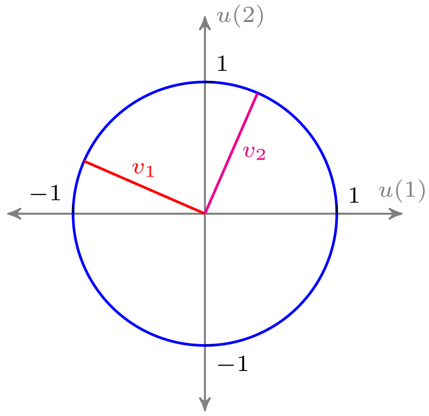
Notation: $\overline{\sigma}(G) := \sigma_1$ (maximum singular value)
$\underline{\sigma}(G) := \sigma_{\min(m,p)}$ (minimum singular value)

Singular value decomposition

Output ($y = Gu$)



Input (u), $\|u\|_2 = 1$



$\sigma_1 = 2.00$ (maximum gain)
$\sigma_2 = 0.82$ (minimum gain)

$$G = \begin{bmatrix} 1.5 & 0 \\ -1.1 & 1.1 \end{bmatrix}$$

SVD properties

Gain bounds for $y = Gu$

For any u, $\quad \underline{\sigma}(G) \leq \frac{\|Gu\|_2}{\|u\|_2} \leq \overline{\sigma}(G)$

Norm relationships:

If $G = G_1 G_2$, $\quad \|G\| = \overline{\sigma}(G) \leq \overline{\sigma}(G_1)\overline{\sigma}(G_2)$

If $G = G_1 + G_2$, $\quad \|G\| = \overline{\sigma}(G) \leq \overline{\sigma}(G_1) + \overline{\sigma}(G_2)$

Condition number

$\kappa(G) := \frac{\overline{\sigma}(G)}{\underline{\sigma}(G)} \qquad$ and by definition: $\kappa(G) \geq 1$

Plants with high condition numbers can be very difficult to control.

Eigenvalue decompositions

Spectral radius

$$\rho(G) = \max_i |\lambda_i(G)|,$$

where $\lambda_i(G)$ satisfies $\lambda_i x_i = G x_i$ (eigenvalue equation).

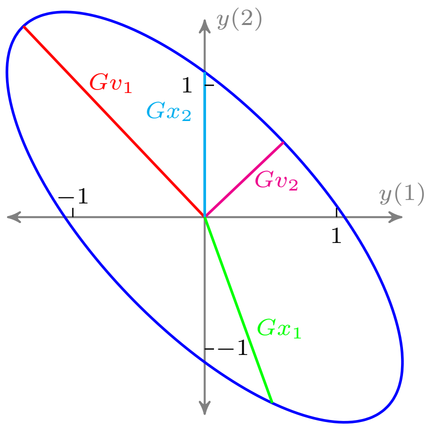
Spectral radius is not a norm

$$G = G_1 + G_2, \quad \text{but} \quad \rho(G) \not\leq \rho(G_1) + \rho(G_2)$$

$$\rho\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 0, \quad \text{but the matrix is not zero.}$$

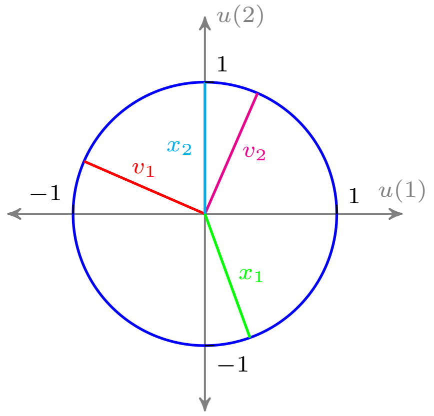
Eigenvalue decomposition

Output ($y = Gu$)



$\lambda_1 = 1.50$
$\lambda_2 = 1.10$

Input (u), $\|u\|_2 = 1$



$$G = \begin{bmatrix} 1.5 & 0 \\ -1.1 & 1.1 \end{bmatrix}$$

Singular value decomposition

Maximum gain; as a function of frequency

$$y(j\omega) \;=\; G(j\omega)\,u(j\omega) \;=\; U(j\omega)\Sigma(j\omega)V(j\omega)^*\,u(j\omega).$$

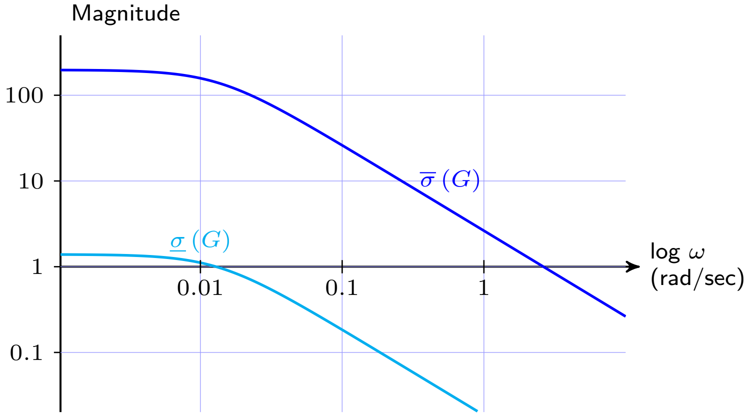
$\mathcal{H}_\infty$ norm

$$\begin{aligned}
\|G(s)\|_{\mathcal{H}_\infty} &= \sup_{\|u(s)\|_2=1} \|y(s)\|_2 \\
&= \sup_{\|u(j\omega)\|_2=1} \frac{\left(\int_{-\infty}^{\infty} y(j\omega)^* y(j\omega) d\omega\right)^{1/2}}{\left(\int_{-\infty}^{\infty} u(j\omega)^* u(j\omega) d\omega\right)^{1/2}} \\
&= \sup_{\omega} \max_{\|u(j\omega)\|_2=1} \|y(j\omega)\|_2 = \sup_{\omega} \overline{\sigma}\left(G(j\omega)\right)
\end{aligned}$$

Example: distillation process (LV configuration)

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}$$

$$\begin{aligned}
\overline{\sigma}\left(G(0)\right) &= 197 \\
\underline{\sigma}\left(G(0)\right) &= 1.39 \\
\kappa(G(0)) &= 141.7
\end{aligned}$$



Distillation process: input errors

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}$$

Open-loop steady state response

Consider the input,

$$u = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{giving the output} \quad y = \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.40 \\ -1.40 \end{bmatrix}.$$

Now suppose we have a 5% error in the input actuators:

$$u = \begin{bmatrix} 1.05 \\ 0.95 \end{bmatrix}. \quad \text{now gives} \quad y = \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}\begin{bmatrix} 1.05 \\ 0.95 \end{bmatrix} = \begin{bmatrix} 10.11 \\ 9.49 \end{bmatrix}.$$

Source of the problem: high condition number; and
input errors not aligned with singular vectors.

Distillation process: model uncertainty

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \quad \approx \quad \tilde{G}(s) = \frac{1}{75s+1}\begin{bmatrix} 87 & -88 \\ 109 & -108 \end{bmatrix}.$$

Suppose we have errors in each matrix entry that are less than 2%.

Open-loop steady state response

Again consider the input,

$$u = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{giving the output} \quad y = \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.40 \\ -1.40 \end{bmatrix}.$$

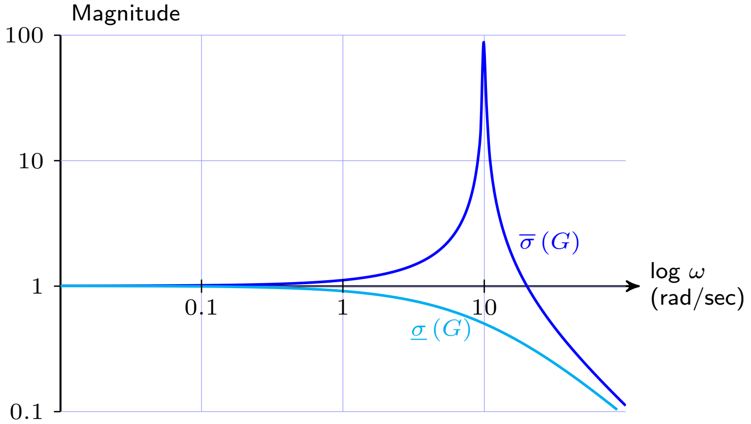
Now apply the same input to $\tilde{G}(0)$,

$$u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad \text{now gives} \quad y = \begin{bmatrix} 87 & -88 \\ 109 & -108 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The sign of each output has changed. Any closed loop controller designed for $\tilde{G}(s)$ is almost certainly unstable on $G(s)$.

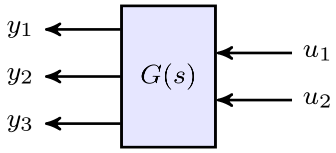
Example: spinning satellite

$$G(s) = \frac{1}{s^2 + a^2} \begin{bmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{bmatrix}, \qquad a = 10.$$



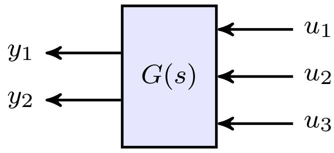
Non-square plants

$n_y > n_u$



There is an output direction which is not affected by the input.

$n_y < n_u$



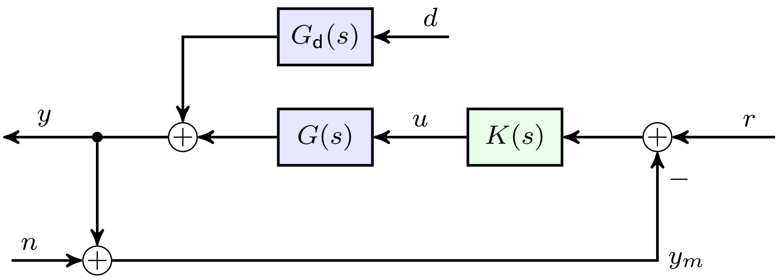
There is an input direction which has no effect on the output.

These directions may vary as a function of frequency.

Functional controllability

A system is functionally controllable if the number of outputs (to be controlled) is equal to the normal rank of $G(s)$.

SVD performance measures



Input gain constraints

Recall the "perfect" control (feedforward) solution:

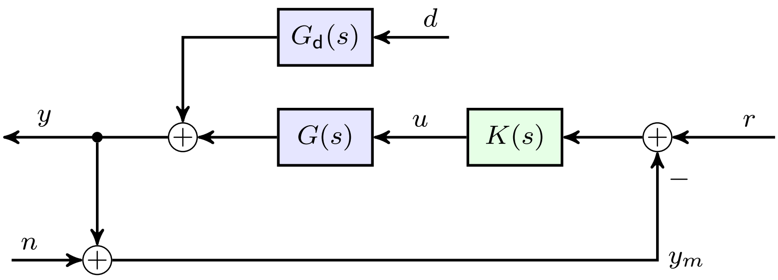
$$u = G(s)^{-1}R\,r - G(s)^{-1}G_d(s)\,d$$

To avoid input saturation:

$$\overline{\sigma}\left(G(j\omega)^{-1}R(\omega)\right) \leq 1 \quad \Leftarrow \quad \underline{\sigma}\left(G(j\omega)\right) \geq \overline{\sigma}\left(R(\omega)\right)$$

$$\overline{\sigma}\left(G(j\omega)^{-1}G_d(j\omega)\right) \leq 1 \quad \Leftarrow \quad \underline{\sigma}\left(G(j\omega)\right) \geq \overline{\sigma}\left(G_d(j\omega)\right)$$

MIMO reference tracking performance



Say $d = 0$, $n = 0$, and $e = r - y$.

$$\underline{\sigma}\left(S_o(j\omega)\right) \leq \frac{\|e(\omega)\|_2}{\|r(\omega)\|_2} \leq \overline{\sigma}\left(S_o(j\omega)\right) \qquad S_o(j\omega) = (I + G(j\omega)K(j\omega))^{-1}$$

To get the reference tracking error, e, small for r of any direction,

$$\overline{\sigma}\left(S_o(j\omega)\right) < 1/|W_p(\omega)|, \quad \text{for all } \omega$$

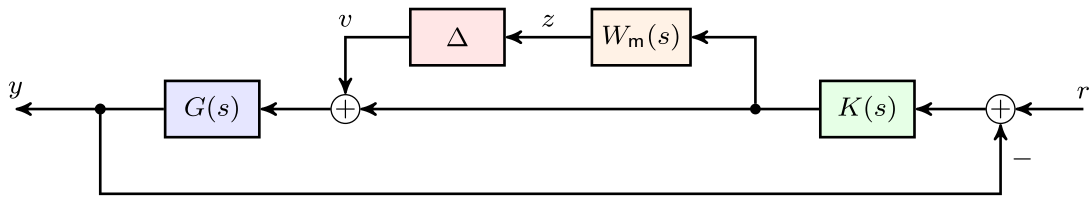
$$\iff \overline{\sigma}\left(W_p(\omega)S_o(j\omega)\right) < 1, \quad \text{for all } \omega$$

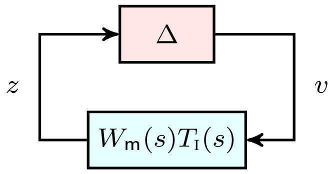
$$\iff \|W_p S_o\|_{\mathcal{H}_\infty} < 1. \qquad \text{(Nominal performance)}$$

Robust stability

Plant input uncertainty

Model set: $\{G(s)(I + \Delta W_\mathsf{m}(\omega)),\quad \|\Delta\|_{\mathcal{H}_\infty} < 1\}$



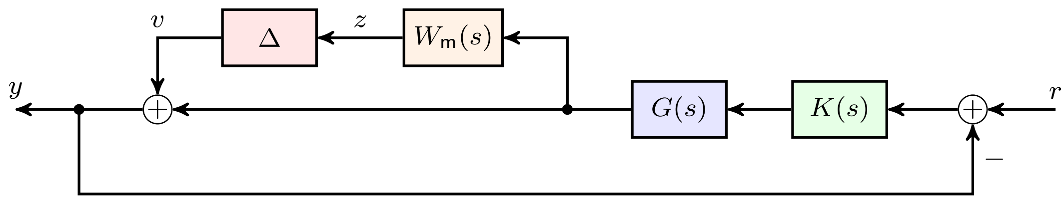


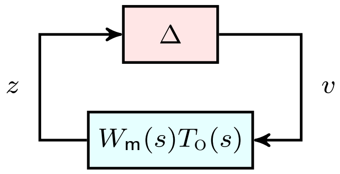
$\|W_\mathsf{m} T_\mathrm{I}\|_{\mathcal{H}_\infty} \le 1 \quad \Longleftrightarrow$ Robust stability

Robust stability

Plant output uncertainty

Model set: $\{(I + \Delta W_\mathsf{m}(\omega))G(s),\quad \|\Delta\|_{\mathcal{H}_\infty} < 1\}$





$\|W_\mathsf{m} T_\mathrm{O}\|_{\mathcal{H}_\infty} \le 1 \quad \Longleftrightarrow$ Robust stability

MIMO loopshaping

Attempt to satisfy $\|W_p S\|_{\mathcal{H}_\infty} < 1$ and $\|W_m T\|_{\mathcal{H}_\infty} \leq 1$.

Nominal performance

$$\frac{1}{\overline{\sigma}(S)} \approx \underline{\sigma}(L) \quad (\text{if } \underline{\sigma}(L) \gg 1)$$

So if $\underline{\sigma}(L) \gg 1$ and $\underline{\sigma}(L) > W_p$ then,

$$\frac{1}{\overline{\sigma}(S)} > W_p \quad \text{implying that} \quad \|W_p S\|_{\mathcal{H}_\infty} < 1.$$

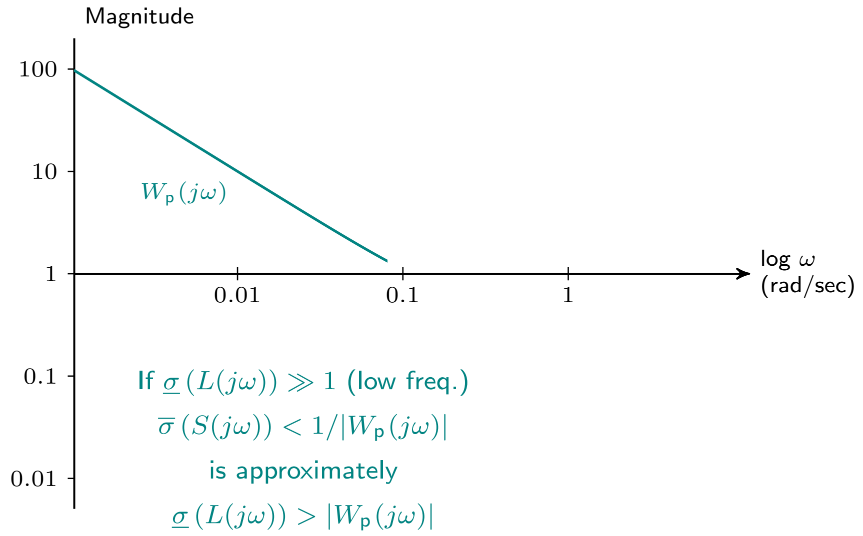
Robust stability

$$\overline{\sigma}(T) \approx \overline{\sigma}(L) \quad (\text{approximately if } \overline{\sigma}(L) \ll 1)$$

So if $\overline{\sigma}(L) \ll 1$ and $\overline{\sigma}(L) < |W_m^{-1}|$ then $\|W_m T\|_{\mathcal{H}_\infty} \leq 1$.

MIMO loopshaping

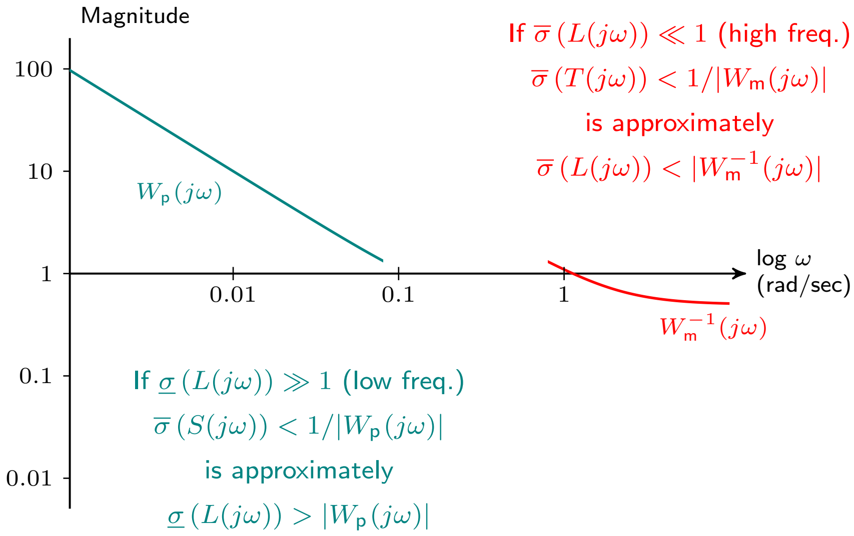
For simplicity, consider scalar weights $W_m(s)$ and $W_p(s)$.



Note: In general $\sigma(GK) \neq \sigma(KG)$. Different conditions may apply.

MIMO loopshaping

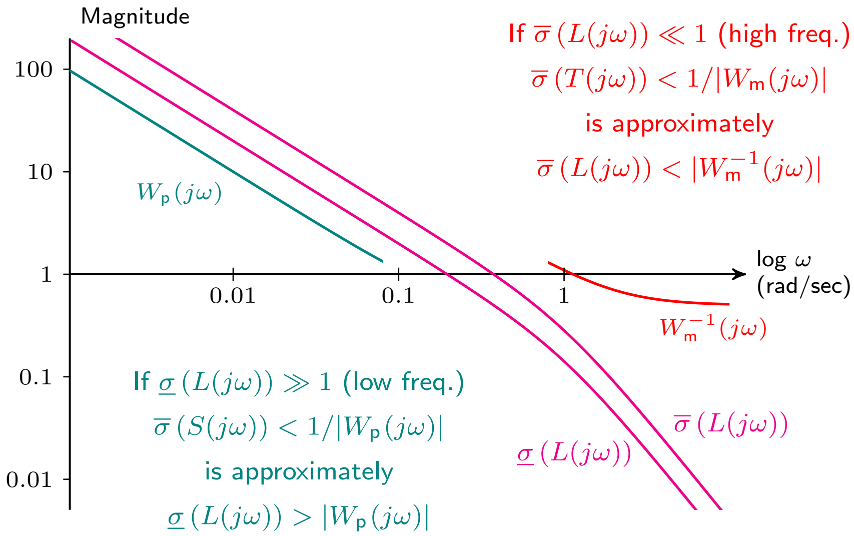
For simplicity, consider scalar weights $W_m(s)$ and $W_p(s)$.



Note: In general $\sigma(GK) \neq \sigma(KG)$. Different conditions may apply.

MIMO loopshaping

For simplicity, consider scalar weights $W_m(s)$ and $W_p(s)$.



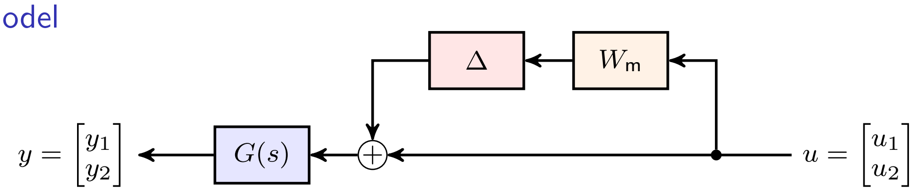
Note: In general $\sigma(GK) \neq \sigma(KG)$. Different conditions may apply.

Example: distillation process (DV configuration)

Nominal plant

$$G(s) = \frac{1}{(100s+1)(s+1)} \begin{bmatrix} -87.8 & 1.4 \\ -108.2 & -1.4 \end{bmatrix}$$

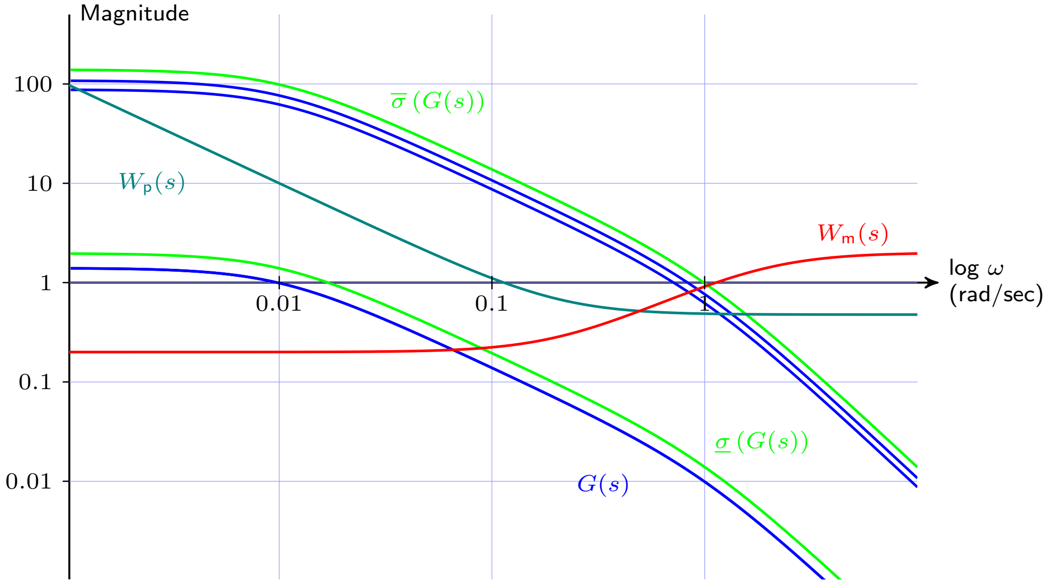
Perturbation model



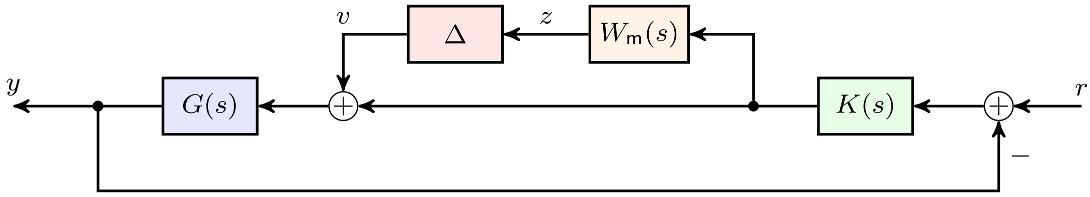
$$W_m(s) = \frac{(s+0.2)}{(0.5s+1)} I_{2\times 2}, \qquad \Delta = \underbrace{\begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}}_{\text{"unstructured" perturbation}} \quad \text{or} \quad \Delta = \underbrace{\begin{bmatrix} \delta_{11} & 0 \\ 0 & \delta_{22} \end{bmatrix}}_{\text{"structured" perturbation}}$$

Example: distillation process (DV configuration)

$$G(s) = \frac{1}{(100s+1)(s+1)} \begin{bmatrix} -87.8 & 1.4 \\ -108.2 & -1.4 \end{bmatrix}, \qquad \kappa(G(j\omega)) = 70.6$$



MIMO controller design: inverse based



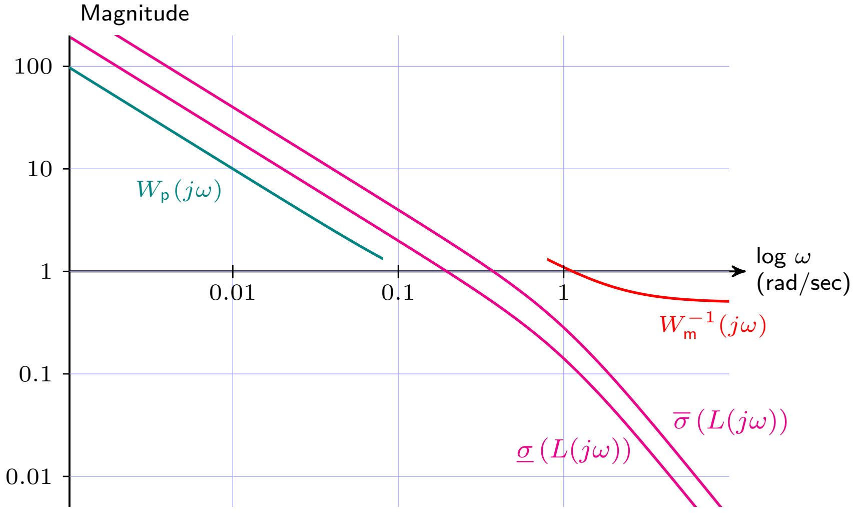
$$G(s) = \frac{1}{(100s+1)(s+1)} \begin{bmatrix} -87.8 & 1.4 \\ -108.2 & -1.4 \end{bmatrix}$$

$$K(s) = \frac{(100s+1)}{s} \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} -87.8 & 1.4 \\ -108.2 & -1.4 \end{bmatrix}^{-1}$$

$$L_I(s) = K(s)G(s) = \frac{1}{s(s+1)} \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix} \quad \longleftarrow \quad \text{RS analysis}$$

$$L_O(s) = G(s)K(s) = \frac{1}{s(s+1)} \begin{bmatrix} 0.29 & 0.09 \\ 0.11 & 0.31 \end{bmatrix} \quad \longleftarrow \quad \text{NP analysis}$$

Example: Singular value loopshapes



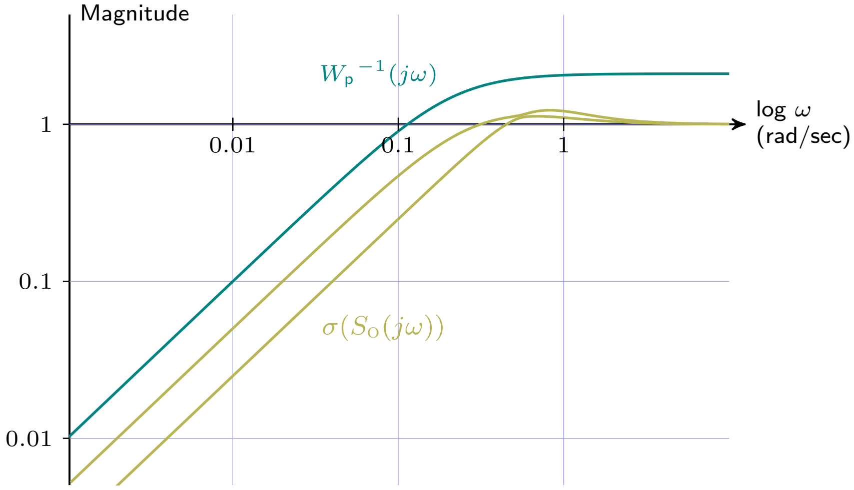
In this case $\sigma(L_I(j\omega)) \approx \sigma(L_O(j\omega))$.

Example: Nominal performance

$$\|W_{\mathsf{p}}S_{\mathsf{O}}\|_{\mathcal{H}_\infty} = \left\|W_{\mathsf{p}}(s)\,(I+G(s)K(s))^{-1}\right\|_{\mathcal{H}_\infty} < 1,$$

or, equivalently,

$$|\overline{\sigma}\,(S_{\mathsf{O}}(j\omega))| < \frac{1}{|W_{\mathsf{p}}(j\omega)|}$$

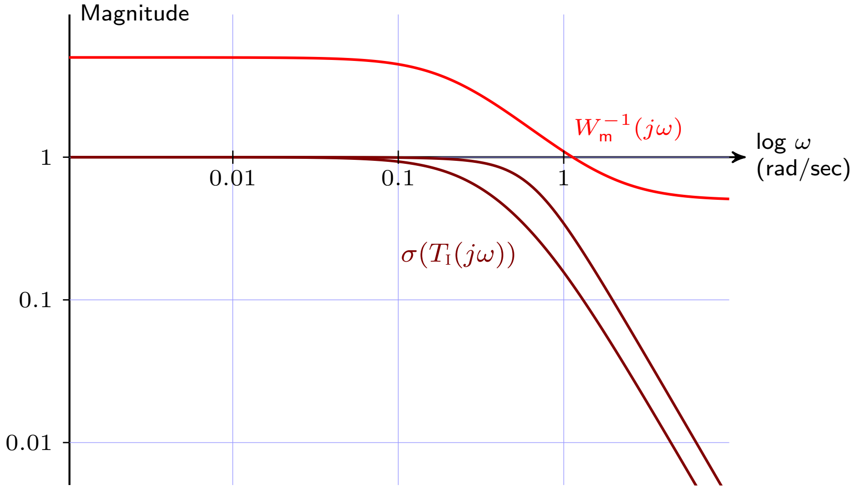


Example: Robust stability

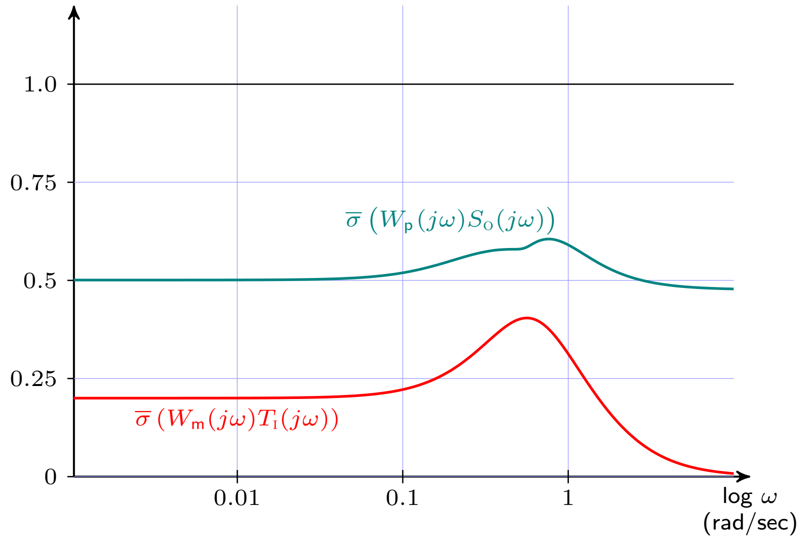
$$\|W_{\mathsf{m}}T_{\mathsf{I}}\|_{\mathcal{H}_\infty} = \left\|W_{\mathsf{m}}(s)\,(I+K(s)G(s))^{-1}K(s)G(s)\right\|_{\mathcal{H}_\infty} < 1,$$

or, equivalently,

$$|\overline{\sigma}\,(T_{\mathsf{I}}(j\omega))| < \frac{1}{|W_{\mathsf{m}}(j\omega)|}$$



Example: nominal performance and robust stability



$\overline{\sigma}\left(W_{\mathsf{p}}(j\omega)S_{\mathrm{O}}(j\omega)\right) < 1$, for all ω $\iff$ nominal performance

$\overline{\sigma}\left(W_{\mathsf{m}}(j\omega)T_{\mathrm{I}}(j\omega)\right) < 1$, for all ω $\iff$ robust stability

MIMO loopshaping

Potential limitations

- Robust stability is with respect to a single unstructured perturbation.
- Ensuring stabilisation is not as simple as the SISO case.
- Multiple performance criteria are not as easily expressed as a single sensitivity constraint.
- No guarantees about robust performance.

MIMO performance limitations

A Bode integral-type sensitivity constraint

$$\int_0^\infty \ln|\det(S(j\omega)|\,d\omega = \sum_j \int_0^\infty \ln\sigma_j(S(j\omega))\,d\omega$$

$$= \begin{cases} \pi \sum_{i=1}^{N_p} \text{real}(p_i) & \text{if } L(s) \text{ is unstable} \\ 0 & \text{if } L(s) \text{ is stable} \end{cases}$$

As in the SISO case, the summation is over the N_p unstable poles of $L(s)$.

MIMO interpolation conditions

Internally stable closed-loop $S(s)$ and $T(s)$

For RHP-poles, p, of $G(s)$, with output direction y_p

$$T(p_i)y_p = y_p, \quad S(p_i)y_p = 0$$

For RHP-zeros, z, of $G(s)$, with output direction y_z

$$y_z^* T(z) = 0, \quad y_z^* S(z) = y_z^*$$

Weighted sensitivity peak

Suppose $G(s)$ has a RHP-zero, z.

Suppose we also have a scalar sensitivity weight, $W_{\mathsf{p}}(s)$.

Then for a closed-loop internally stable sensitivity function, $S(s)$,

$$\begin{aligned}\|W_{\mathsf{p}}(s)S(s)\|_{\mathcal{H}_\infty} &= \max_{\omega} \overline{\sigma}\left(W_{\mathsf{p}}(j\omega)S(j\omega)\right) \\ &\geq |W_{\mathsf{p}}(z)|\end{aligned}$$

Performance requirements

$$\|W_{\mathsf{p}}(s)S(s)\|_{\mathcal{H}_\infty} < 1 \quad \Longrightarrow \quad |W_{\mathsf{p}}(z)| < 1.$$

Weighted complementary sensitivity peak

Suppose $G(s)$ has a RHP-pole, p.

Suppose we also have a scalar perturbation weight, $W_{\mathsf{m}}(s)$.

Then for a closed-loop internally stable sensitivity function, $T(s)$,

$$\begin{aligned}\|W_{\mathsf{m}}(s)T(s)\|_{\mathcal{H}_\infty} &= \max_{\omega} \overline{\sigma}\left(W_{\mathsf{m}}(j\omega)T(j\omega)\right) \\ &\geq |W_{\mathsf{m}}(p)|\end{aligned}$$

Robustness requirements

$$\|W_{\mathsf{m}}(s)T(s)\|_{\mathcal{H}_\infty} < 1 \quad \Longrightarrow \quad |W_{\mathsf{m}}(p)| < 1.$$

MIMO RHP zero limitations

Non-minimum phase MIMO example

$$G_{\mathsf{NMP}}(s) = \begin{bmatrix} \dfrac{s}{s^2+11s+10} & \dfrac{5s^2+10s+50}{s^3+15s^2+50s} \\ \dfrac{10}{s^2+11s+10} & \dfrac{s+55}{s^2+15s+50} \end{bmatrix}$$

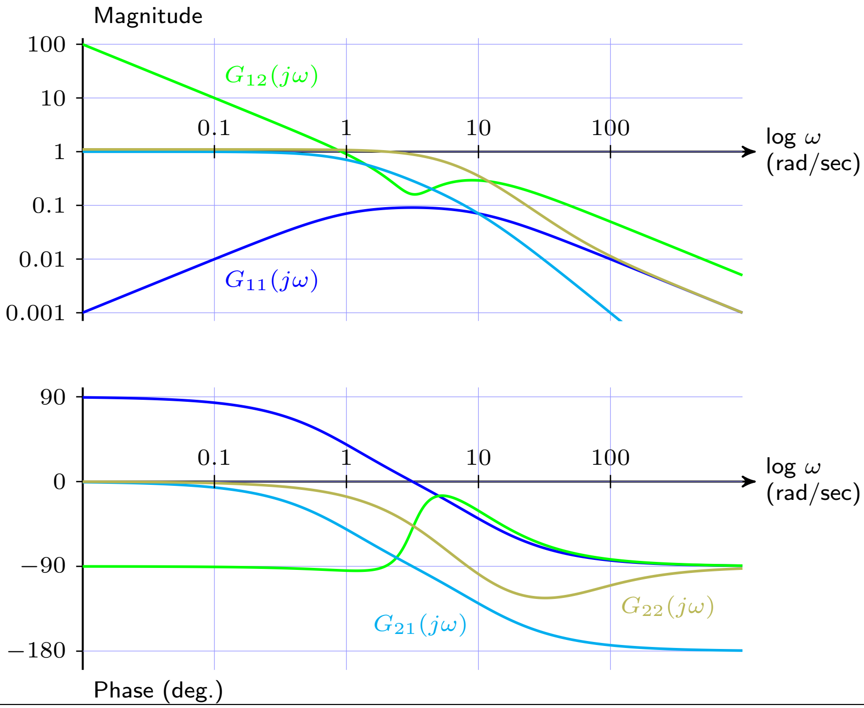
$$= \underbrace{\begin{bmatrix} \dfrac{s}{s+10} & \dfrac{10}{s+10} \\ \dfrac{10}{s+10} & \dfrac{s}{s+10} \end{bmatrix}}_{B_z(s)^{-1}} \underbrace{\begin{bmatrix} \dfrac{1}{s+1} & \dfrac{5}{s+5} \\ 0 & \dfrac{1}{s} \end{bmatrix}}_{G_{\mathsf{MP}}(s)} .$$

$G_{\mathsf{NMP}}(s)$ has a RHP zero at $s = 10$ radians/second.

$$\underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{y_z^T} \; G_{\mathsf{NMP}}(s) \,|_{s=10} = \begin{bmatrix} 0 & 0 \end{bmatrix} .$$

MIMO RHP zero limitations

Non-minimum phase MIMO example



MIMO RHP zero limitations

Internal Model Control (IMC) based design

Design for,

$$T(s) \;=\; B_z^{-1}(s) T_{\text{ideal}}(s).$$

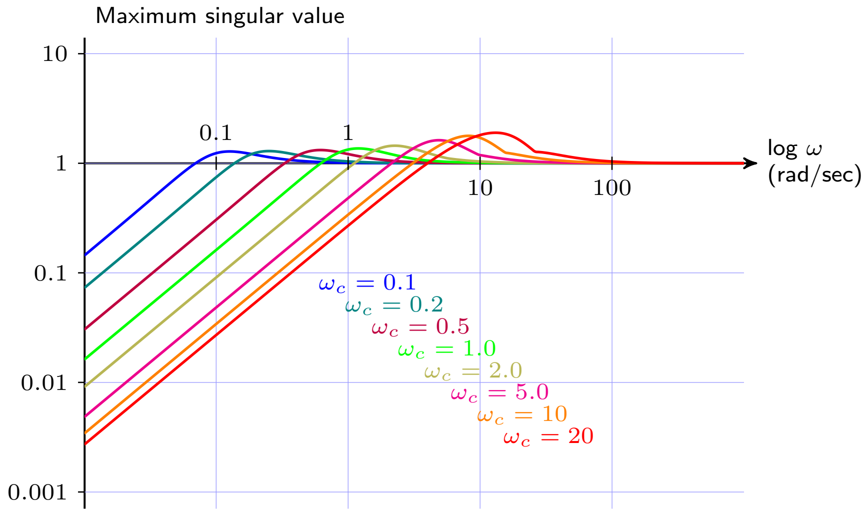
Choose the "ideal" complementary sensitivity function to be,

$$T_{\text{ideal}}(s) = \frac{1}{\left(\frac{s}{\omega_c}\right)^2 + \sqrt{2}\left(\frac{s}{\omega_c}\right) + 1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

MIMO RHP zero limitations

Sensitivity: $S = (I + GK)^{-1}$

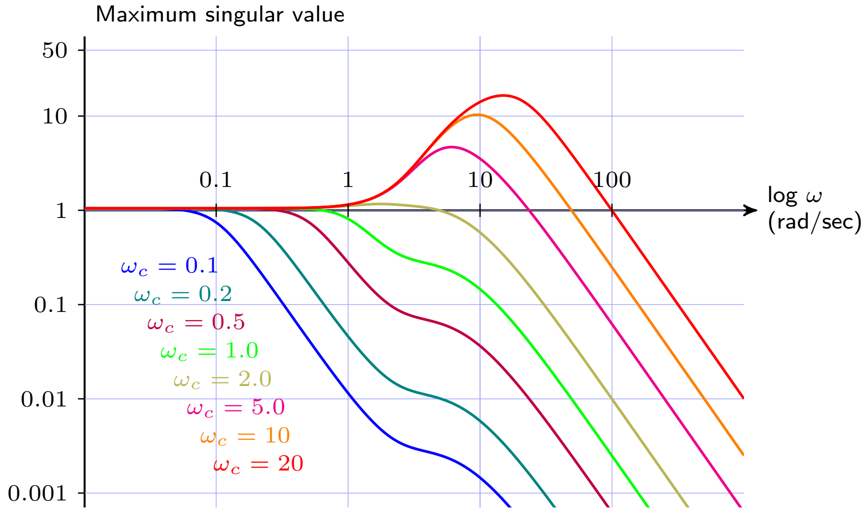
Reference tracking performance or output disturbance rejection.



MIMO RHP zero limitations

Input complementary sensitivity: $T_i = (I + KG)^{-1}$

Robustness with respect to input perturbations.



Notes and references

Skogestad & Postlethwaite (2nd Ed.)

MIMO systems: sections 3.1 – 3.3

MIMO design & examples: sections 3.5 – 3.7, & 9.1

Sensitivity bounds: sections 6.1 – 6.3