Control Systems 2

Lecture 6: Uncertainty and robustness in SISO systems

Roy Smith

Uncertainty

Who is uncertain and what are they uncertain about?

Sources of uncertainty

- Design to be mass produced (many plants);
- Plant aging;
- Unmodeled dynamics (too complex, no easy model);
- Neglected dynamics (accuracy is too expensive);
- Operation over a range of operating points;
- Non-repeatable dynamic behaviour.

Robustness

Nominal stability (NS) Is the closed-loop system stable when the plant is known exactly?

Robust stability (RS) Is the closed-loop system stable when there is uncertainty in our knowledge of the plant?

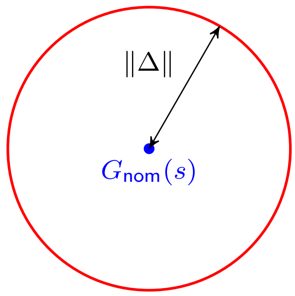
Nominal performance (NP) Does the closed-loop system meet the performance specifications when the plant is known exactly?

Robust performance (RP) Does the closed-loop system meet the performance specifications when there is uncertainty in our knowledge of the plant?

Uncertain models

Model sets

$$G(s) \in \{\, G_{\text{nom}}(s) + \Delta \mid \|\Delta\| \leq \gamma \,\}$$



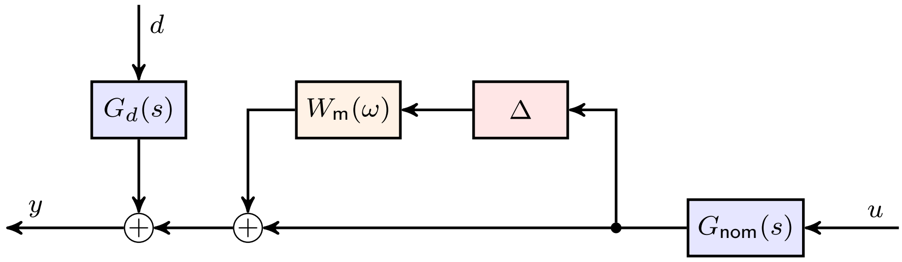
$G_{\text{nom}}(s)$ = Nominal plant

Δ = unknown, but bounded, perturbation (i/o operator)

Typically, Δ is stable, causal and satisfies, $\|\Delta\|_{\mathcal{H}_\infty} \leq \gamma$.

Uncertain models

Multiplicative perturbation model

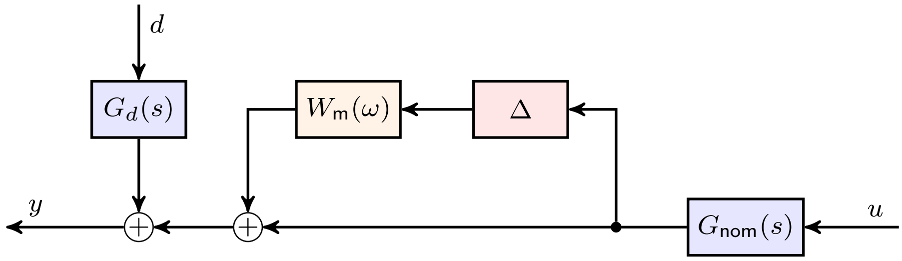


$y(s) = G(s)u(s) + G_d(s)d(s)$, where $G(s) \in \mathcal{G}$,

$\mathcal{G} = \{ (1 + W_m(\omega)\Delta)G_{nom}(s) \mid \Delta$ is causal, stable; $\|\Delta\|_{\mathcal{H}_\infty} \leq 1 \}$.

Uncertain models

Multiplicative perturbation model



$$y = (I + W_m(\omega)\Delta)G_{nom}\,u + G_d\,d$$

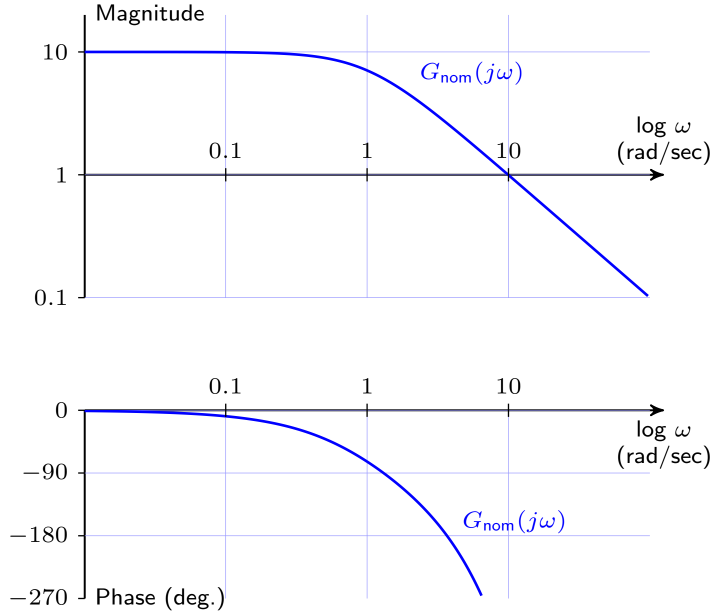
$$= \underbrace{G_{nom}\,u}_{\text{nominal response}} + \underbrace{W_m(\omega)\Delta G_{nom}\,u}_{\text{perturbation response}} + \underbrace{G_d\,d}_{\text{disturbance response}}$$

Example: uncertain model (first order plus delay)

$$G = \frac{K e^{-\lambda s}}{1 + \tau s}$$

Nominal case

$$K = 10$$
$$\tau = 1.0$$
$$\lambda = 0.5$$

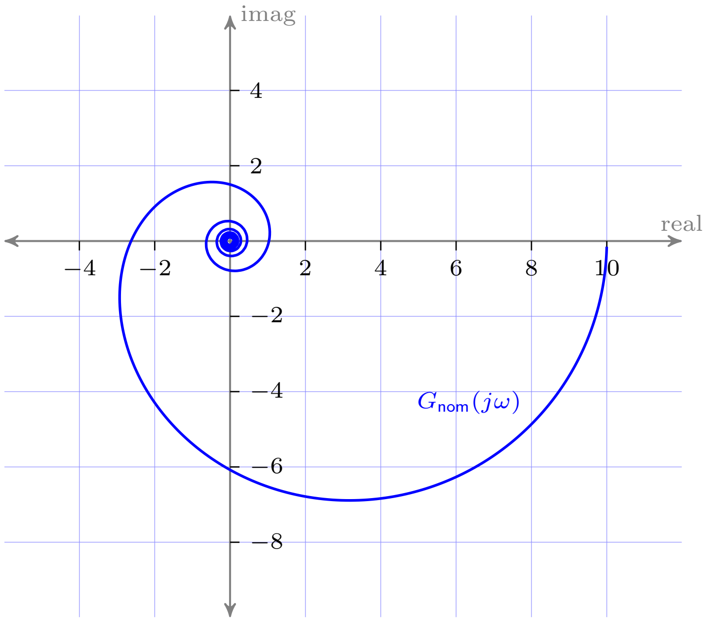


Example: uncertain model (first order plus delay)

$$G = \frac{K e^{-\lambda s}}{1 + \tau s}$$

Nominal case

$$K = 10$$
$$\tau = 1.0$$
$$\lambda = 0.5$$



Example: uncertain model (first order plus delay)

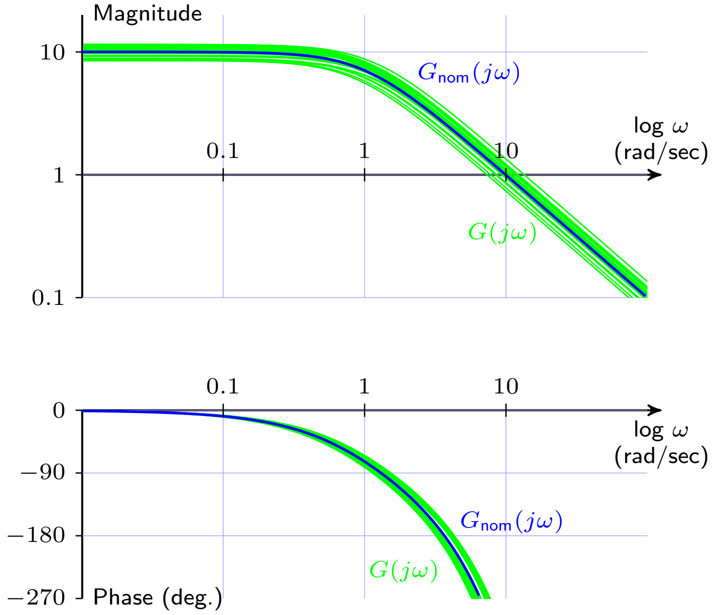
Random examples

$$G = \frac{K e^{-\lambda s}}{1 + \tau s}$$

Perturbed case
$\pm 15\%$ uncertainty

$K \in [8.5, 11.5]$
$\tau \in [0.85, 1.15]$
$\lambda \in [0.425, 0.575]$

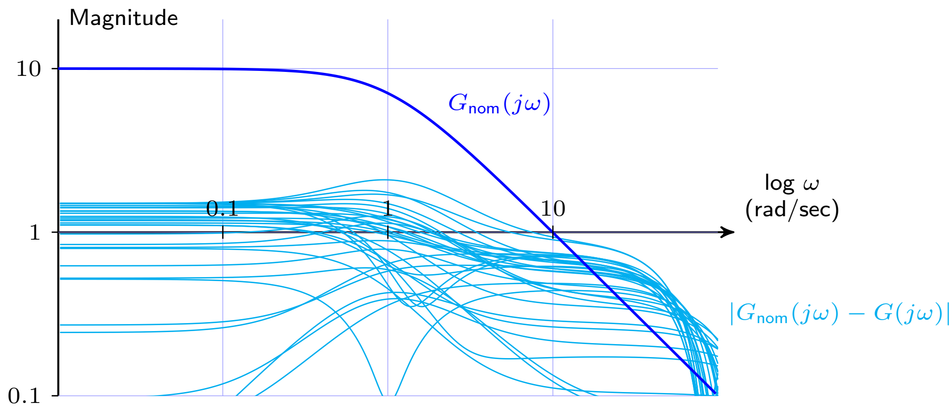
also includes boundary cases.



Example: uncertain model (first order plus delay)

Absolute error (random examples)

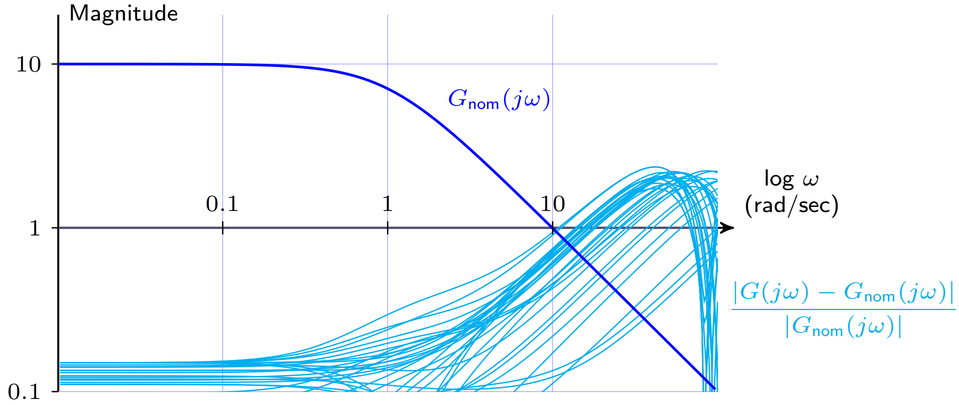
Error as a function of frequency: $|G_{\text{nom}}(j\omega) - G(j\omega)|$



Example: uncertain model (first order plus delay)

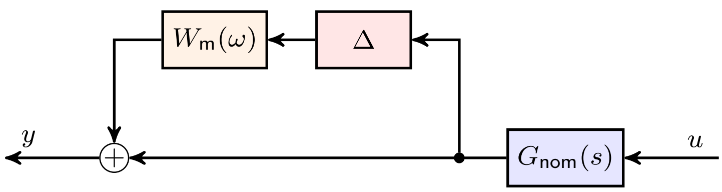
Relative error (random examples)

Error as a function of frequency: $\dfrac{|G_{\text{nom}}(j\omega) - G_{(}j\omega)|}{|G_{\text{nom}}(j\omega)|}$



Example: uncertain model (first order plus delay)

Modeling problem: choose G_{nom} and W_{m}



Upper bound on relative uncertainty: choosing $W_{\text{m}}(s)$

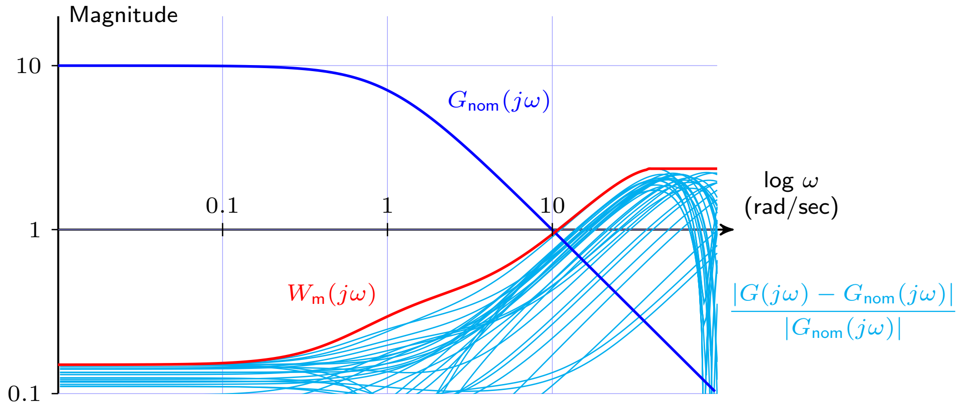
$$G(s) = (1 + W_{\text{m}}(\omega)\Delta)G_{\text{nom}}(s) \implies \frac{G(j\omega) - G_{\text{nom}}(j\omega)}{G_{\text{nom}}(j\omega)} = W_{\text{m}}(\omega)\Delta$$

so $$\left|\frac{G(j\omega) - G_{\text{nom}}(j\omega)}{G_{\text{nom}}(j\omega)}\right| \le |W_{\text{m}}(\omega)| \implies |\Delta| \le 1.$$

Example: uncertain model (first order plus delay)

Upper bound on relative uncertainty: choosing $W_{\text{m}}(s)$

$$\left|\frac{G(j\omega) - G_{\text{nom}}(j\omega)}{G_{\text{nom}}(j\omega)}\right| \leq |W_{\text{m}}(\omega)|.$$

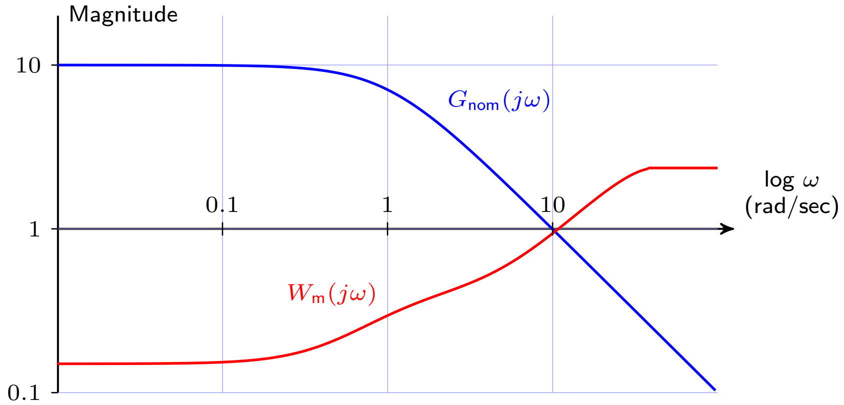


Example: uncertain model (first order plus delay)

$$G(s) = \left\{ \frac{K\text{e}^{-\lambda s}}{1+\tau s} \;\middle|\; K = 10,\ \tau = 1,\ \lambda = 0.5,\ \text{(all } \pm 15\%\text{)} \right\}$$

$$G(s) \in \left\{ (1 + W_{\text{m}}(s)\Delta)G_{\text{nom}}(s) \;\middle|\; \|\Delta\|_{\mathcal{H}_\infty} \leq 1 \right\},$$

where here, $W_{\text{m}}(s) = \dfrac{1.15(1+\tau s)}{(1+0.85\tau s)}\text{e}^{0.15\lambda s} - 1$ (up to its maximum, and then constant)

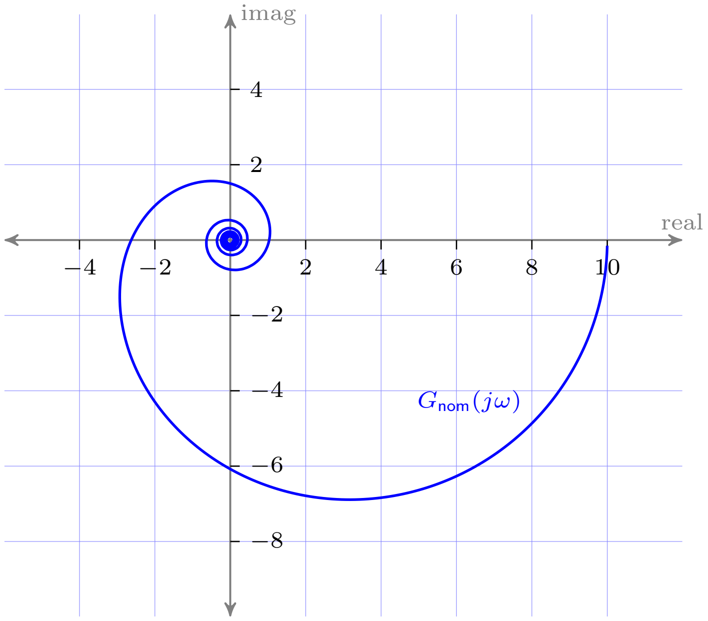


(See Laughlin *et al.* in reference list)

Example: uncertain model (first order plus delay)

Nominal system

$$G = \frac{K\mathrm{e}^{-\lambda s}}{1+\tau s}$$



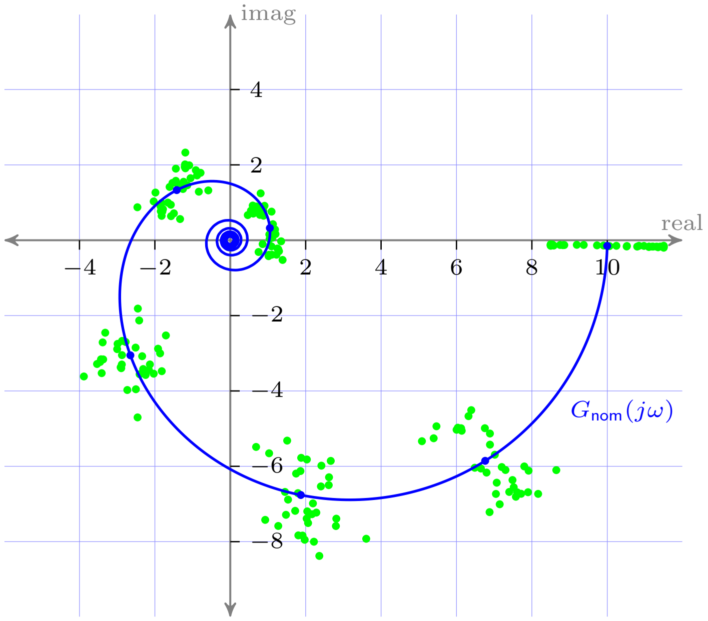
Example: uncertain model (first order plus delay)

Random examples

$$G = \frac{K\mathrm{e}^{-\lambda s}}{1+\tau s}$$

Perturbations: $\pm 15\%$

on K, λ, τ.



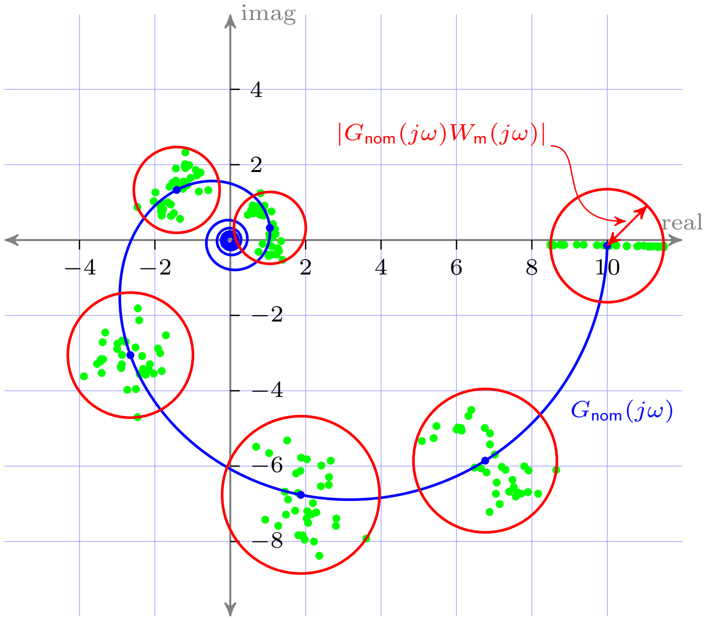
Example: uncertain model (first order plus delay)

Random examples

$$G = \frac{K\mathrm{e}^{-\lambda s}}{1+\tau s}$$

Perturbations: $\pm 15\%$

on K, λ, τ.



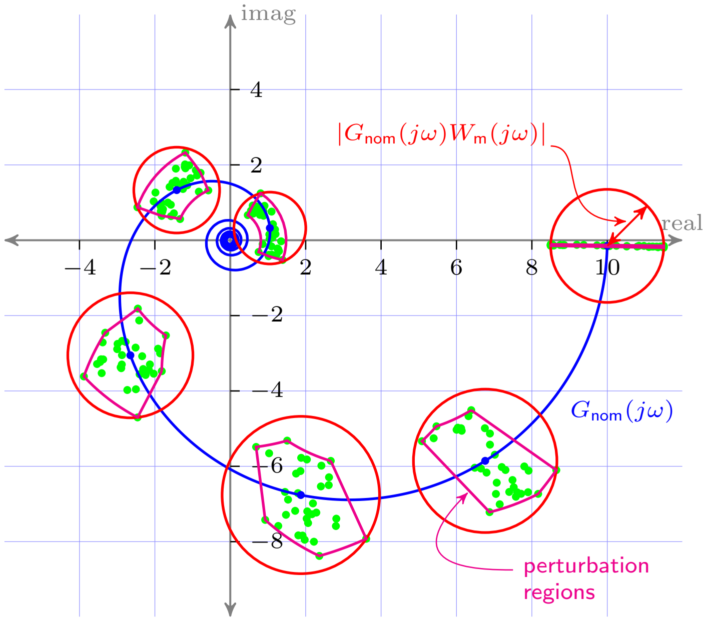
Example: uncertain model (first order plus delay)

Random examples

$$G = \frac{K\mathrm{e}^{-\lambda s}}{1+\tau s}$$

Perturbations: $\pm 15\%$

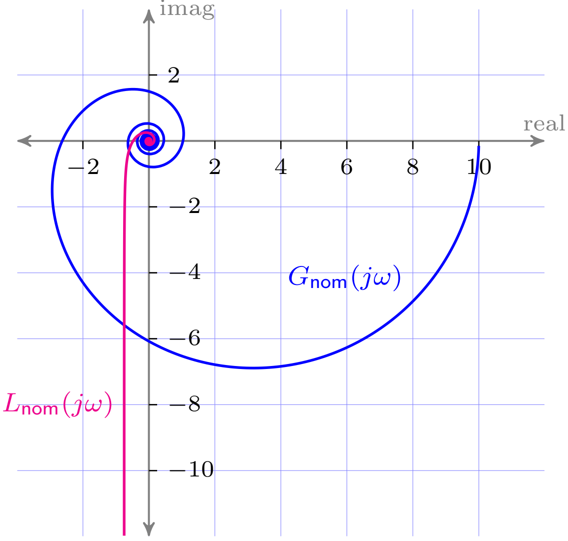
on K, λ, τ.

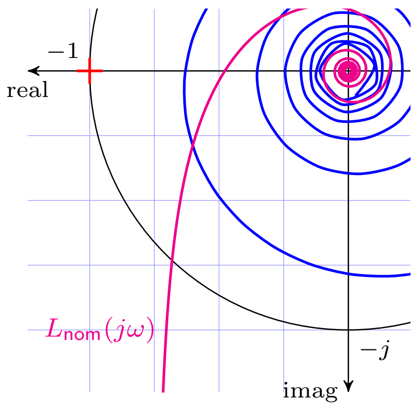


Design example: Loopshaping design

$$G_{\mathsf{nom}}(s) = \frac{K\mathrm{e}^{-\lambda s}}{1+\tau s}, \qquad K = 10, \quad \tau = 1, \quad \lambda = 0.5$$

$$K(s) = \frac{0.15(1+\tau s)}{s} \qquad \text{and} \qquad L_{\mathsf{nom}}(s) = \frac{0.15K\mathrm{e}^{-\lambda s}}{s}$$

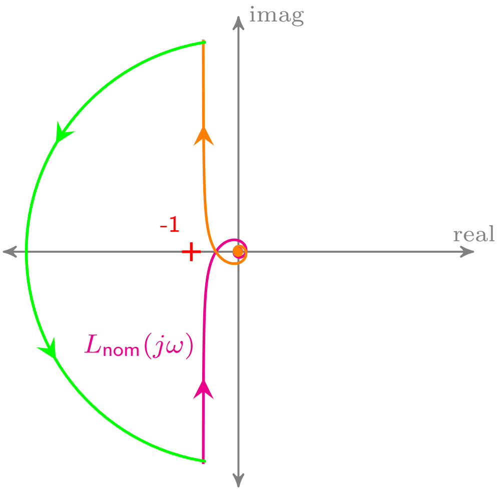


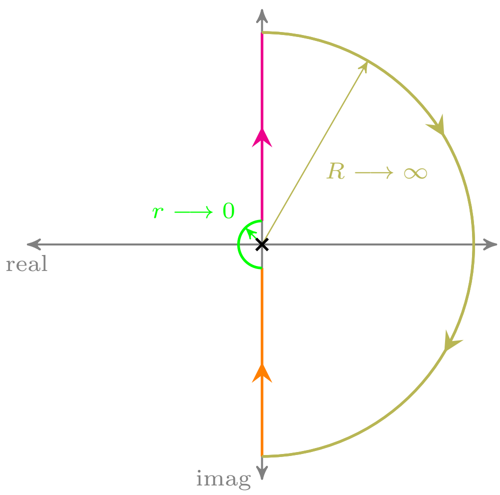


Nominal stability

Nyquist criterion in more detail

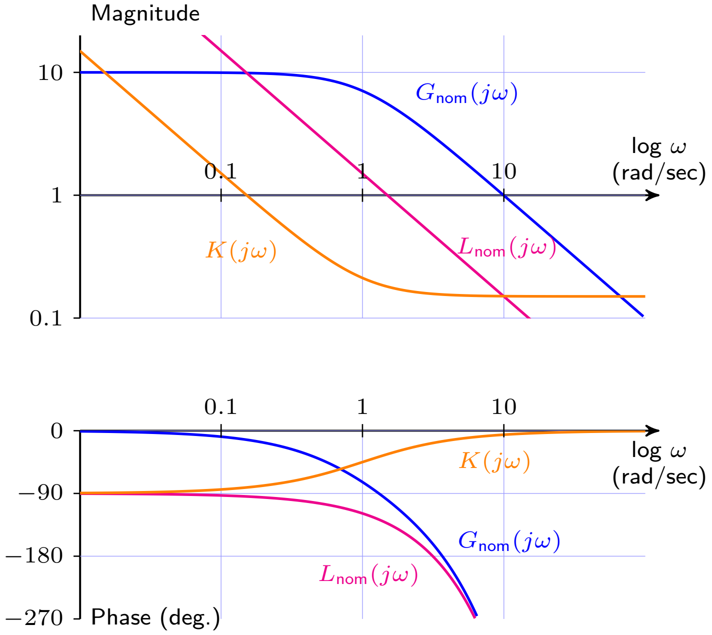
$$L_{\mathsf{nom}}(s) = \frac{0.15K\mathrm{e}^{-\lambda s}}{s}$$





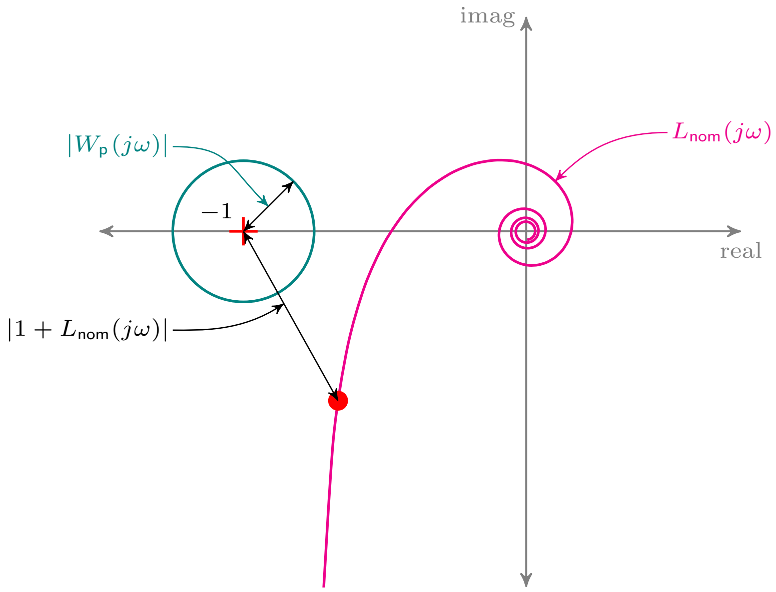
Nominal stability example: Loopshaping design

$$G_{\mathsf{nom}}(s) = \frac{K\mathrm{e}^{-\lambda s}}{1+\tau s}, \qquad K(s) = \frac{0.15(1+\tau s)}{s}, \qquad L_{\mathsf{nom}}(s) = \frac{0.15K\mathrm{e}^{-\lambda s}}{s}$$



Nominal performance

$$|W_{\mathsf{p}}(j\omega)S_{\mathsf{nom}}(j\omega)| < 1 \qquad \text{for all } \omega.$$

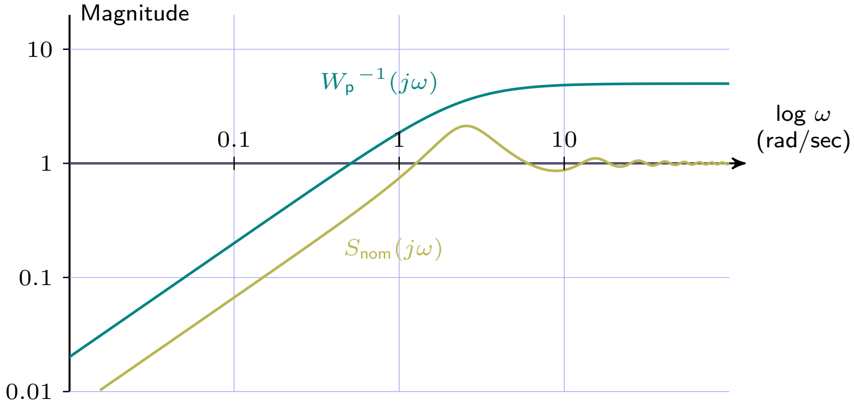


Nominal performance

Nominal weighted sensitivity: $S_{\text{nom}}(s) = \dfrac{1}{1 + G_{\text{nom}}(s)K(s)}$

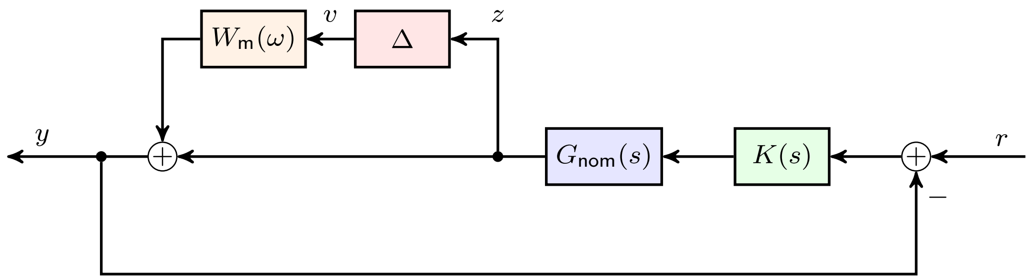
Nominal performance $\iff$ $|S_{\text{nom}}(j\omega)| < \dfrac{1}{|W_{\text{p}}(j\omega)|}$

In the loopshaping example: $W_{\text{p}}(s) = \dfrac{s + 2.5}{5s}$.



Robust stability

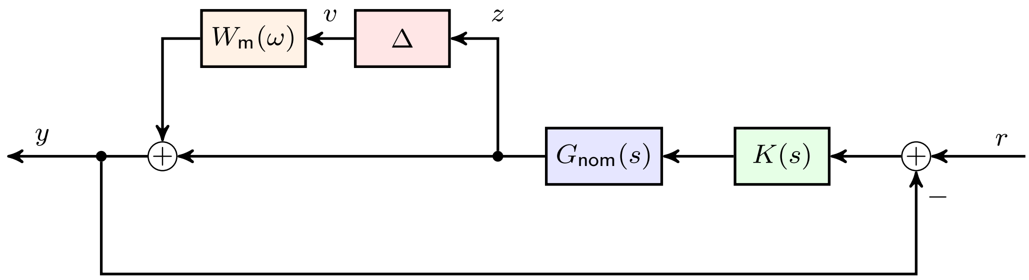
Closed-loop multiplicative perturbation model



Robust stability

Is this closed-loop stable for all stable, causal Δ, with $\|\Delta\|_{\mathcal{H}_\infty} \leq 1$?

Robust stability

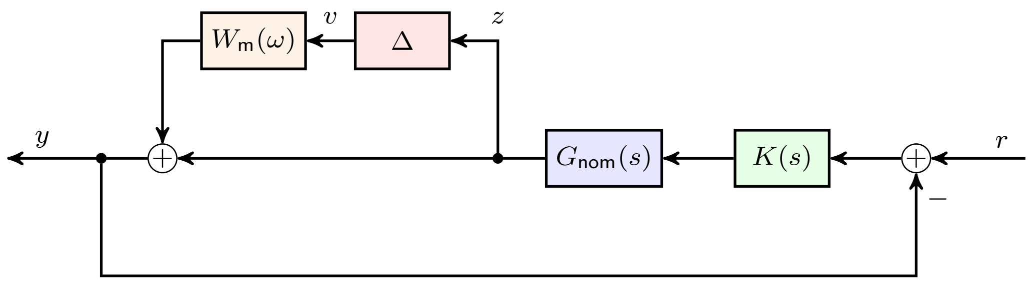


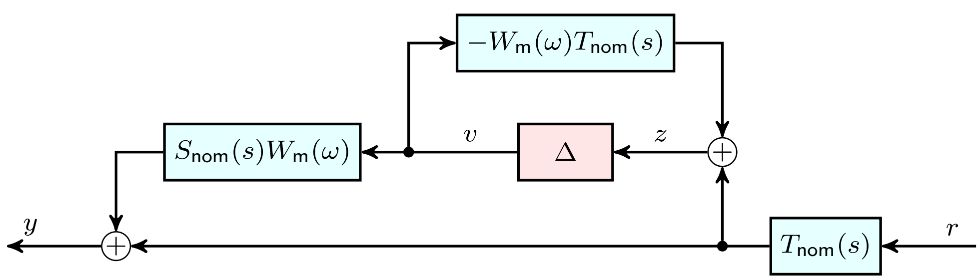
Robust stability

$$S(s) = \frac{1}{1 + G(s)K(s)} = \frac{1}{1 + (1 + W_m(s)\Delta)G_{nom}(s)K(s)}$$

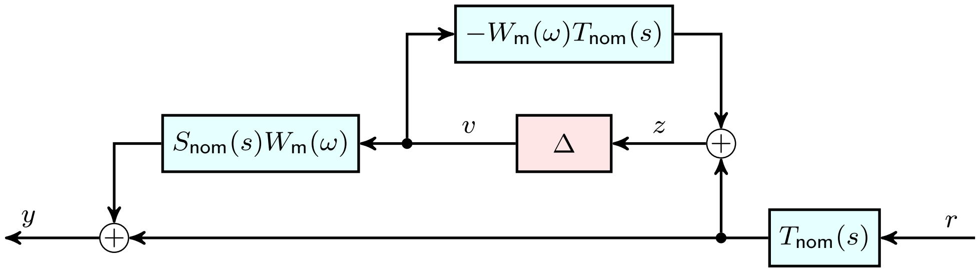
is stable for all Δ, $\quad \|\Delta\|_{\mathcal{H}_\infty} \leq 1.$

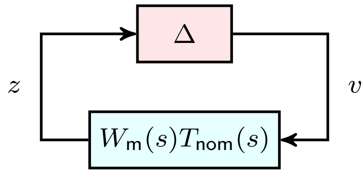
Robust stability



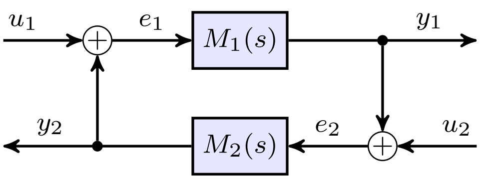


Robust stability: an equivalent test





Small gain theorem



Given e_1, e_2 with $\|e_1\| < \infty$ and $\|e_2\| < \infty$, define,

$$u_1 = e_1 - M_2(s)e_2 \quad \text{and} \quad u_2 = e_2 - M_1(s)e_1.$$

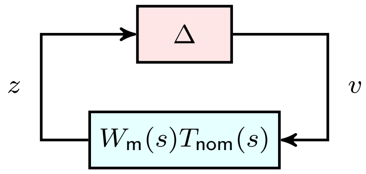
Suppose there exists $\gamma_1 > 0$ and $\gamma_2 > 0$ such that,

$$\|M_1(s)e_1\| \leq \gamma_1\|e_1\| \quad \text{and} \quad \|M_2(s)e_2\| \leq \gamma_2\|e_2\|.$$

If $\gamma_1\gamma_2 < 1$ then,

$$\|y_1\| \leq \frac{\gamma_1}{1 - \gamma_1\gamma_2}(\|u_1\| + \gamma_2\|u_2\|).$$

Robust stability: an equivalent test



By applying the small-gain theorem:

If $\quad \|\Delta\|\|W_m(s)T_{nom}(s)\| < 1$

then the perturbed closed-loop system is stable.

Or:

If $\quad \|W_m(s)T_{nom}(s)\|_{\mathcal{H}_\infty} < 1$

then the perturbed closed-loop system is stable (RS).

Robust stability

Robust stability $\iff$ $\dfrac{1}{1+G(s)K(s)}$ is stable for all $G(s) \in \mathcal{G}$.

$\iff$ $\dfrac{1}{1+(1+W_m(s)\Delta)G_{nom}(s)K(s)}$ is stable for all $\|\Delta\|_{\mathcal{H}_\infty} \leq 1$

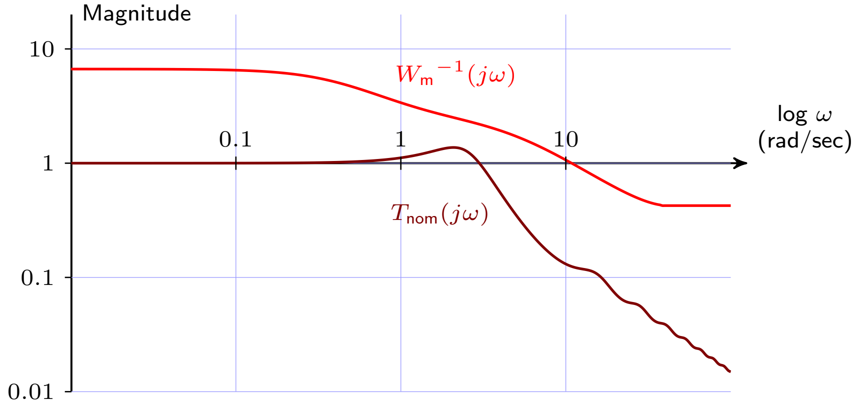
$\iff$ $\|W_m(s)T_{nom}(s)\|_{\mathcal{H}_\infty} < 1$

$\iff$ $|W_m(j\omega)T_{nom}(j\omega)| < 1 \quad$ for all ω

$\iff$ $|T_{nom}(j\omega)| < \dfrac{1}{|W_m(j\omega)|} \quad$ for all ω.

Robust stability: Loopshaping design example

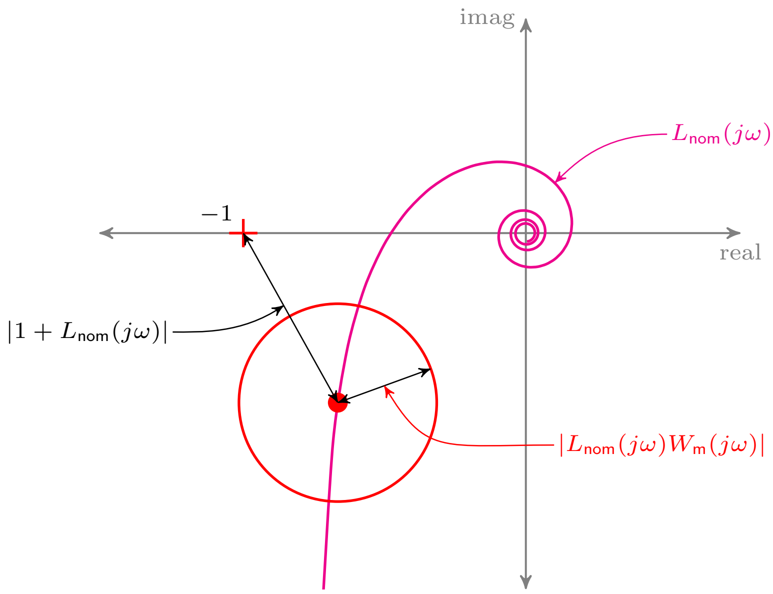
Robust stability $\iff$ $|T_{\text{nom}}(j\omega)| < \dfrac{1}{|W_{\text{m}}(j\omega)|}$ for all ω.



Robust stability

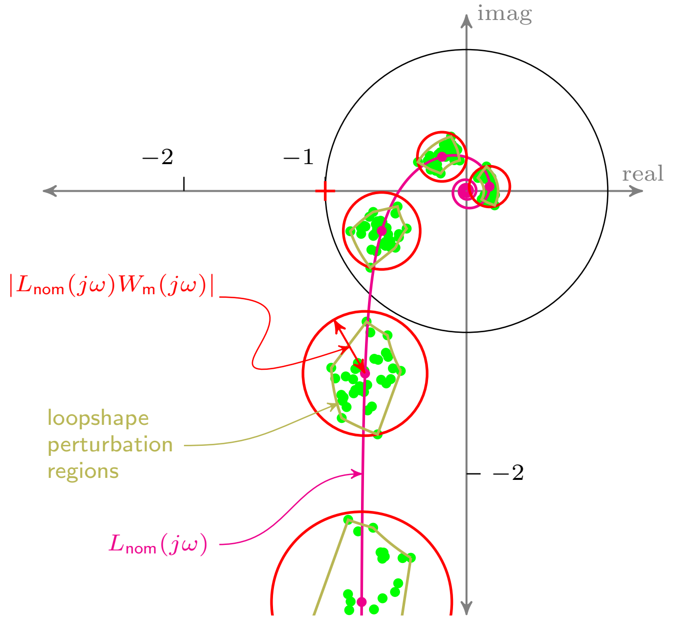
$$|1 + G_{\text{nom}}(j\omega)K(j\omega)| > |W_{\text{m}}(\omega)G_{\text{nom}}(j\omega)K(j\omega)|$$

$$\iff |T_{\text{nom}}(j\omega)| < \frac{1}{|W_{\text{m}}(j\omega)|} \quad \text{for all } \omega.$$



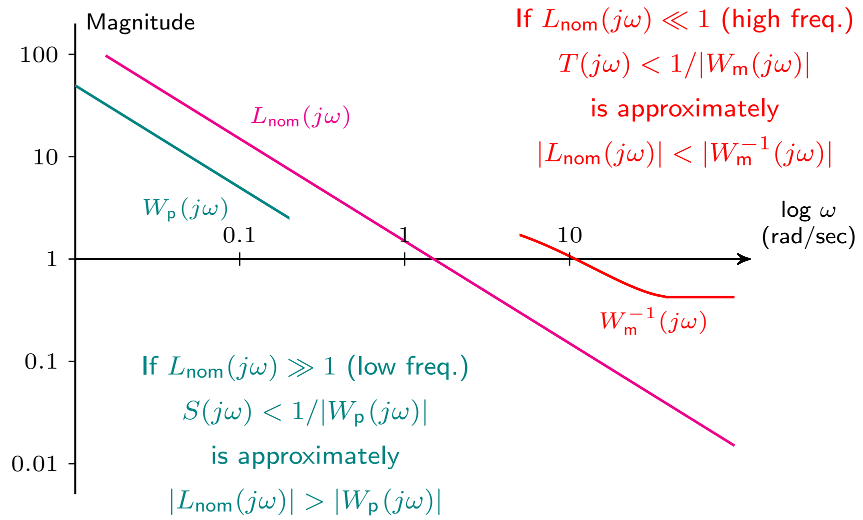
Example: uncertain model (first order plus delay)

Nyquist criterion: perturbed loopshape regions



Loopshaping approximations

Nominal performance and robust stability



Robust performance

Robust performance $\iff$ $\left\| \frac{W_{\mathsf{p}}(s)}{1+G(s)K(s)} \right\|_{\mathcal{H}_\infty} < 1$, for all $G(s) \in \mathcal{G}$.

$\iff$ $\left\| \frac{W_{\mathsf{p}}(s)}{1+(1+W_{\mathsf{m}}(s)\Delta)G_{\mathsf{nom}}(s)K(s)} \right\|_{\mathcal{H}_\infty} < 1$,

for all $\|\Delta\|_{\mathcal{H}_\infty} \leq 1$

$\iff$ $\frac{|W_{\mathsf{p}}(j\omega)|}{|1+(1+W_{\mathsf{m}}(j\omega)\Delta)G_{\mathsf{nom}}(j\omega)K(j\omega)|} < 1$,

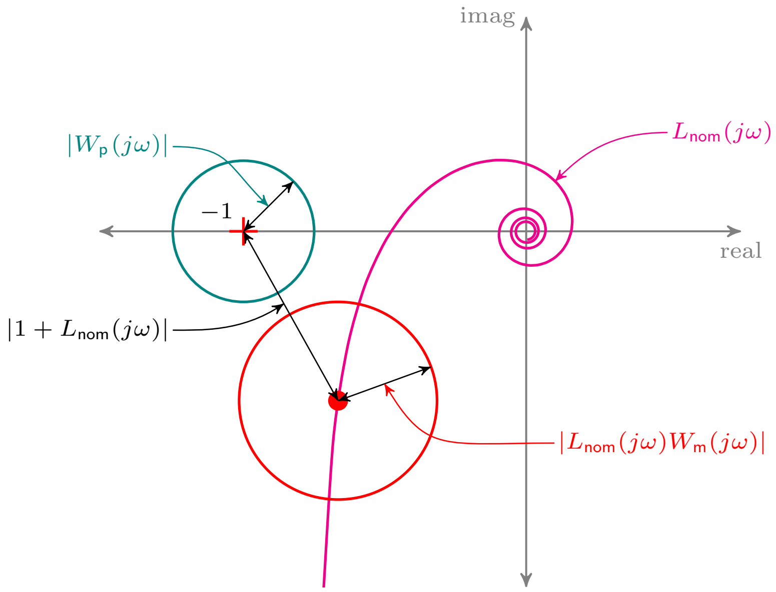
for all ω and all $|\Delta| \leq 1$

which is equivalent to,

$$|W_{\mathsf{p}}(j\omega)S_{\mathsf{nom}}(j\omega)| + |W_{\mathsf{m}}(j\omega)T_{\mathsf{nom}}(j\omega)| < 1 \quad \text{for all } \omega.$$

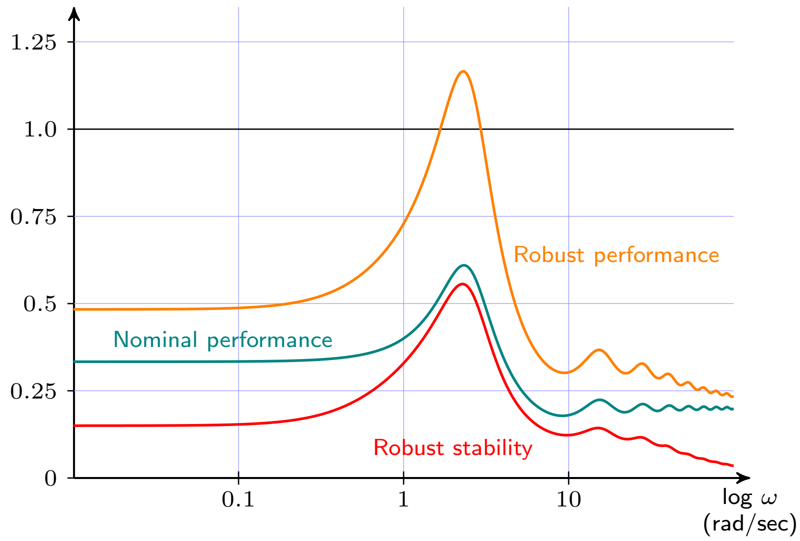
Robust performance

$$|W_{\mathsf{p}}(j\omega)S_{\mathsf{nom}}(j\omega)| + |W_{\mathsf{m}}(j\omega)T_{\mathsf{nom}}(j\omega)| < 1 \quad \text{for all } \omega.$$



Robust performance example

$$|W_p(j\omega)S_{nom}(j\omega)| + |W_m(j\omega)T_{nom}(j\omega)| < 1 \quad \text{for all } \omega.$$



Notes and references

Skogestad & Postlethwaite (2nd Ed.)

Pertubation models: sections 7.1, 7.2, 7.3, 7.4

SISO robust stability: section 7.5

SISO robust performance: section 7.6

Performance limitations due to uncertainty: section 5.12

Perturbation bound, $W_m(s)$, for the RS example

"Smith predictor design for robust performance," DL Laughlin, DE Rivera, M Morari, *Int. J. Control,* v. 46, no. 2, pp. 477–504, 1987.