Exercise 12 - Reachability, Observability and Order Reduction

12.1 Reachability and Observability

We continue our system analysis with two properties of particular interest for control-related tasks: reachability and observability. Informally, reachability relates to how well one can force a system towards some desired point of its state space, while observability has to do with how well one can reconstruct the initial condition (and thus the entire trajectory) of a system by only looking at its output.
Mathematically, the problem can be stated as follows: given an LTI system

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{12.1}$$

with $x(t) \in \mathbb{R}^n$, we define the following conceptual questions:

Reachability Given a system as in (12.1), what points $\bar{x} \in \mathbb{R}^n$ can be reached in finite time by applying inputs u of finite energy and when starting at $x(0) = 0$?

Observability Given a system as in (12.1), what initial conditions $x(0) \in \mathbb{R}^n$ can be reconstructed using only the outputs $y(t)$?

We now present a set of tools that enable us to answer these questions.

12.1.1 Reachability

A point $\bar{x} \in \mathbb{R}^n$ is said to be reachable if, starting at $x(0) = 0$, there exists a $\tau < \infty$ such that $x(\tau) = \bar{x}$. Furthermore, if the latter statement is true for all possible $\bar{x}$, the system $\{A, B\}$ is completely reachable.
Complete reachability is guaranteed if and only if the matrix

$$\mathcal{R}_n = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

is full rank, i.e., if $\text{rank}(\mathcal{R}_n) = n$.
Remark. For LTI systems the concepts of reachability and controllability are equivalent.

12.1.2 Observability

A system $\{A, C\}$ is said to be completely observable if, given $y(t)$ up to some time τ, that is, given the function $y(t)$ for $t \in [0, \tau]$, the initial condition $x(0)$ can be reconstructed uniquely. The latter is true if the matrix

$$\mathcal{O}_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is full rank, i.e., if $\text{rank}(\mathcal{O}_n) = n$.

This material is based on the HS17 teaching assistance taught by Nicolas Lanzetti and Gioele Zardini.
This document can be downloaded at `https://n.ethz.ch/~lnicolas`

Remark. Note that by reconstructing $x(0)$ we can actually recover the whole trajectory $x(t)$.

Remark. This theory is applicable to any linear system, regardless of whether the system is essentially linear or it is the result of a linearization procedure.

12.2 Gramian Matrices and Order Reduction

The above criteria provide a binary answer on full controllability and observability. What happens, however, when the reachability/observability matrices are very close to being rank deficient? Is there a way to understand how reachable/observable a system is? In the following section, we investigate a quantitative measure for these important properties.

Example 1. Consider the system

$$A = \begin{bmatrix} -1 & a \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The reachability and observability matrices read

$$\mathcal{R}_2 = \begin{bmatrix} 0 & a \\ 1 & -1 \end{bmatrix} \qquad \mathcal{O}_2 = \begin{bmatrix} 1 & 0 \\ -1 & a \end{bmatrix}.$$

For any $a \neq 0$ both $\mathcal{R}_2$ and $\mathcal{O}_2$ are full ($n = 2$) rank, and the test we proposed above succeeds in assessing full reachability and observability respectively.
But what happens if $0 < a \ll 1$? Is there a way to assess quantitatively *how* reachable/observable a system is?

12.2.1 Gramian Matrices

The so-called controllability and observability Gramians for linear time-invariant systems are defined by

$$\begin{aligned} W_R &= \int_0^\infty e^{At} B B^\top e^{A^\top t} \,\mathrm{d}t, \\ W_O &= \int_0^\infty e^{A^\top t} C^\top C e^{At} \,\mathrm{d}t. \end{aligned} \tag{12.2}$$

The closer these matrices are to being singular, the less reachable and controllable the system is respectively. Computing W_R and W_O with the definition is quite cumbersome, however if A is Hurwiz, W_R and W_O exist and are solution to the Lyapunov equations

$$\begin{aligned} A W_R + W_R A^\top &= -B B^\top, \\ A^\top W_O + W_O A &= -C^\top C, \end{aligned} \tag{12.3}$$

In MATLAB these equations can be solved with the `lyap` command as follows

```
W_R = lyap(A,B*B');
W_O = lyap(A',C'*C);
```

Remark. The Gramians are, by construction, symmetric and positive definite.

12.2.2 Order Reduction

Is it possible to interpret the information provided by the Gramians and figure out which states are poorly reachable/observable? In general, W_R and W_O are non-diagonal matrices and it can be difficult to extract information from their entries. However, by introducing an appropriate "balanced" transformation

$$x = Tx_b$$

such that

$$W_{R,b} = W_{O,b} = \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \qquad \text{with } \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n,$$

we can solve this problem.
The transformation matrix T can be found in a systematic way as follows:

1. $W_R = V_R \Lambda_R^2 V_R^\top \quad \Longrightarrow \quad T_R = V_R \Lambda_R$

   In MATLAB: `[V_R,D_R] = eig(W_R)` and `Lambda_R = sqrt(D_R)`.

2. $\tilde{W}_O = T_R^\top W_O T_R = V_O \Lambda_O^2 V_O^\top \quad \Longrightarrow \quad T_O = V_O \Lambda_O^{-\frac{1}{2}}$

   In MATLAB: `[V_O,D_O] = eig(T_R'*W_O*T_R)` and `Lambda_O = sqrt(D_O)`.

3. $T = T_R \cdot T_O \quad \Longrightarrow \quad W_{R,b} = W_{O,b} = T^\top W_O T = T^{-1} W_R T^{-\top} = \Lambda_O$

4. $A_b = T^{-1}AT, \quad B_b = T^{-1}B, \quad C_b = CT, \quad D_b = D.$

Remark. It can happen that one has to permute the entries of the Gramians to achieve the desired descending magnitudes of σ_i.

The system can then be partitioned as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$
$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du$$

where $x_1 \in \mathbb{R}^{n-\nu}$ and $x_2 \in \mathbb{R}^\nu$ and all matrices are of appropriate dimensions.
To achieve a better reachable/observable system we have two options.

Approach 1: Simply ignore the influence of the last ν states. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1 = A_{11}x_1 + B_1 u$$
$$y = C_1 x_1 + Du.$$

This procedure normally offers good performances. However, the DC gain of the original system and of the reduced one are often different.

Approach 2 (single perturbation analysis): Ignore the dynamics of the last ν states, but not their contribution to the system at steady state. Then, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}x_2 = 0 \quad \Rightarrow \quad 0 = A_{21}x_1 + A_{22}x_2 + B_2 u \quad \Rightarrow \quad x_2 = -A_{22}^{-1}(A_{21}x_1 + B_2 u).$$

The dynamics of x_1 result in

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x_1 &= A_{11}x_1 + A_{12}x_2 + B_1u = (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ y &= C_1x_1 + C_2x_2 + Du = (C_1 - C_2A_{22}^{-1}A_{21})x_1 + (D - C_2A_{22}^{-1}B_2)u.\end{aligned}$$

This approach does not change the DC gain.

Remark. Note that if the system is asymptotically stable, then the matrix A_{22} is guaranteed to be invertible.

12.3 Tips

a.ii) To check whether a matrix A is Hurwitz without computing its eigenvalues, one can use the so-called Hurwitz criterion (see Wikipedia entry on Routh-Hurwitz criterion) applied on the characteristic polynomial.

c.iii) Given an LTI in standard form, the steady-state gain can be computed as

$$\frac{y_0}{u_0} = (-cA^{-1}b + d).$$

Example

In order to optimize your spaghetti production, you want to reduce the order of a linear system with the matrices

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ -0.5 & -2 & -1 \end{bmatrix}, \; b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; c = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \; d = 0$$

from 3 to 2.

1. Compute the observability and reachability matrices. What can you conclude the about the reachability and observability of the system?
2. Give a sufficient condition for the existence of Gramian matrices.
3. Show that the Gramian matrices exist.
4. Explain the necessary steps to reduce the order of the system.
5. After point 3., you get the following balanced system

   $$\bar{A} = \begin{bmatrix} -1.9 & -0.06 & -0.08 \\ -0.06 & -0.04 & -1.15 \\ 0.08 & 1.15 & -0.06 \end{bmatrix}, \; \bar{b} = \begin{bmatrix} 0.41 \\ 0.46 \\ 0.61 \end{bmatrix}, \; \bar{c} = \begin{bmatrix} -0.41 & -0.46 & 0.61 \end{bmatrix}, \; \bar{d} = 0.$$

   The observability and controllability Gramians are given as

   $$W_{R,b} = W_{O,b} = \begin{bmatrix} 0.04 & 0 & 0 \\ 0 & 2.93 & 0 \\ 0 & 0 & 3.17 \end{bmatrix}.$$

   Which state would you eliminate in order to reduce the order of the system from 3 to 2? Why?
6. Give the system matrices of the reduced system.
7. Has the DC gain changed?
8. Show how you would overcome this issue.

Solution. 1. Computing the reachability (controllability) and observability matrices results in

$$\mathcal{R}_3 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0.5 \end{bmatrix} \qquad \mathcal{O}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -0.5 & -2 & -1 \end{bmatrix}$$

We can immediately see that $\text{rank}(\mathcal{O}_3) = \text{rank}(\mathcal{R}_3) = 3$ since the respective column vectors are linearly independent. We can therefore conclude that the system is fully reachable/controllable and fully observable.

2. The condition is that the system is asymptotically stable.

3. We need to show that the condition is fulfilled. The fastest way to show this, is to prove that the matrix A is Hurwitz, i.e., all of its eigenvalues have negative real part. This can be investigated e.g. by looking at all the leading principal minors of H (the Hurwitz matrix of the characteristic polynomial of A): if they are all positive, the matrix is Hurwitz. It holds

$$\begin{aligned}\det(A - \lambda \mathbb{I}) &= \det \begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -1-\lambda & 0 \\ -0.5 & -2 & -1-\lambda \end{bmatrix} \\ &= -\lambda \cdot (\lambda+1)^2 - 2 - \frac{1}{2} \cdot (\lambda + 1) \\ &= \lambda^3 + 2\lambda^2 + \frac{3}{2}\lambda + \frac{5}{2} = 0.\end{aligned}$$

From here one has two choices:

- You compute the eigenvalues and you show that these have negative real part. Here

$$\begin{aligned}\lambda_1 &= -0.0488 + 1.1453i \\ \lambda_2 &= -0.0488 - 1.1453i \\ \lambda_3 &= -1.9023.\end{aligned}$$

- You compute the Hurwitz matrix of the characteristic polynomial of A. For a generic polynomial

$$p(x) = a_0 x^3 + a_1 x^2 + a_2 x + a_3$$

this reads

$$\begin{bmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{bmatrix}.$$

For our case we have

$$H = \begin{bmatrix} 2 & 2.5 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 2.5 \end{bmatrix}$$

$$\det(2) = 2 > 0$$

$$\det \begin{bmatrix} 2 & 2.5 \\ 1 & 2 \end{bmatrix} = 1.5 > 0$$

$$\det(H) = 2.5 \cdot 1.5 = \frac{15}{4} > 0.$$

4. To reduce the order of the system one should:
   - Calculate the Gramians for controllability and observability by solving the two Lypunov equations.
   - Find the coordinate transformation matrix T, which transforms the original system into a system whose controllability and observability Gramians are identical and diagonal (see Section 12.2.2).
   - Apply the coordinate transformation to derive the balanced system $\{T^{-1}AT, T^{-1}b, cT, d\}$.
5. The first state has a two order smaller influence on the controllability and observability of the system. For this reason we can eliminate it.
6. The reduced system reads

$$\tilde{A} = \begin{bmatrix} -0.04 & -1.15 \\ 1.15 & -0.06 \end{bmatrix}, \ \tilde{b} = \begin{bmatrix} 0.46 \\ 0.61 \end{bmatrix}, \ \tilde{c} = \begin{bmatrix} -0.46 & 0.61 \end{bmatrix}, \ \tilde{d} = 0.$$

7. In steady state conditions (using $u = 1$), one can write

$$\dot{x} = Ax + b = 0,$$

which has the solution

$$x_{\text{ss}} = \begin{bmatrix} 0.4 \\ 0.4 \\ 0 \end{bmatrix}.$$

For this reason it holds

$$y_{\text{ss}} = x_1 = 0.4.$$

For the same input $u = 1$, one can compute

$$\begin{aligned} -0.04\tilde{x}_2 - 1.15\tilde{x}_3 &= -0.46 \\ 1.15\tilde{x}_2 - 0.06\tilde{x}_3 &= -0.61. \end{aligned}$$

This system has solution

$$\tilde{x}_{\text{ss,red}} = \begin{bmatrix} -0.51 \\ 0.42 \end{bmatrix}.$$

It follows

$$y_{\text{ss,red}} = -0.46\tilde{x}_2 + 0.61\tilde{x}_3 = 0.49,$$

which is higher than the one for the original system.

8. In order to solve this problem, you should use the singular perturbation method. In this method, the eliminated state is computed stationary and dynamically with the other state variables. According to the balanced system matrix, the differential equation of the state which has to be eliminated is

$$\dot{\tilde{x}}_1 = -1.9\tilde{x}_1 - 0.06\tilde{x}_2 - 0.08\tilde{x}_3 + 0.41\tilde{u}$$

By setting the derivative to 0, one gets

$$\tilde{x}_1 = -0.03\tilde{x}_2 - 0.04\tilde{x}_3 + 0.22\tilde{u}$$