Exercise 10 - System Analysis

10.1 Autonomous and Non-Autonomous Systems

Consider a nonlinear continuous-time system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = f(x(t), t) \qquad \text{with } x(t_0) = x_0, \tag{10.1}$$

where $x(t) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Such systems are called non-autonomous or time-varying (TV). If the system features no explicit time dependency, it is called autonomous or time-invariant (TI).
Note that TV systems can be transformed in TI form by augmenting the state space as follows. Let

$$\tilde{x}(t) = \begin{bmatrix} x(t) \\ x_{n+1}(t) \end{bmatrix}$$

with $x_{n+1}(t) = t$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x}(t) = \begin{bmatrix} f(x(t), x_{n+1}(t)) \\ 1 \end{bmatrix} =: \tilde{f}(\tilde{x}(t)) \qquad \text{with } \tilde{x}(t_0) = \begin{bmatrix} x_0 \\ t_0 \end{bmatrix}.$$

10.2 Asymptotic Behavior of Systems

Definition 1 (Limit Point)**.** A point $p \in \mathbb{R}^n$ is a limit point of the system $\dot{x} = f(x, t)$ if there exists a sequence of points $x(t_1), x(t_2), \ldots$ with the same initial condition $x(t_0) = x_0$ such that

$$\lim_{i\to\infty} \begin{bmatrix} t_i \\ x(t_i) \end{bmatrix} = \begin{bmatrix} \pm\infty \\ p \end{bmatrix} \qquad \text{for } x_0 \neq p.$$

Definition 2 (Limit Set)**.** A limit set is set of limit points with the same initial condition $x(t_0) = x_0$.

Definition 3 (Attractor and Repellor)**.** A limit set is called

- *attractor* if it is approached for $t_i \to +\infty$;
- *repellor* if it is approached for $t_i \to -\infty$.

There are four types of limit sets:

Equilibria: An equilibrium x^* is a point in $\mathbb{R}^n$ for which $f(x^*, t) = 0\, \forall t$.

Periodic Solutions: A periodic solution is a trajectory for which it holds that $x(t+T) = x(t)\, \forall t$, where T is the period of the solution. For TV systems: $f(x, t) = f(x, t + \tau)$ with $k\tau = T$, where τ is the period of the system.

Remark. A continuous-time time-invariant system can exhibit periodic solutions only if $n \geq 2$.

Quasiperiodic Solutions: A quasiperiodic solution is characterized by two more incommensurate frequencies, i.e. $\omega/\Omega \in \mathbb{R} \setminus \mathbb{Q}$.

Chaos: Chaos in a deterministic dynamical system is bounded steady state behaviour that is not an equilibrium, periodic solution or quasiperiodic solution. Chaos is aperiodic oscillation with a sensitive dependence on initial condition.

Remark. A continuous-time time-invariant system can exhibit chaotic solutions only if $n \geq 3$.

10.3 Lyapunov Stability

Definition 4 (Lyapunov Stable Equilibrium)**.** An equilibrium point x^* of $\dot{x} = f(x)$ is Lyapunov stable if

$$\forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 : \quad \|x(t_0) - x^*\| < \delta \Longrightarrow \|x(t) - x^*\| < \varepsilon \quad \forall t \geq t_0.$$

Otherwise, x^* is called unstable.

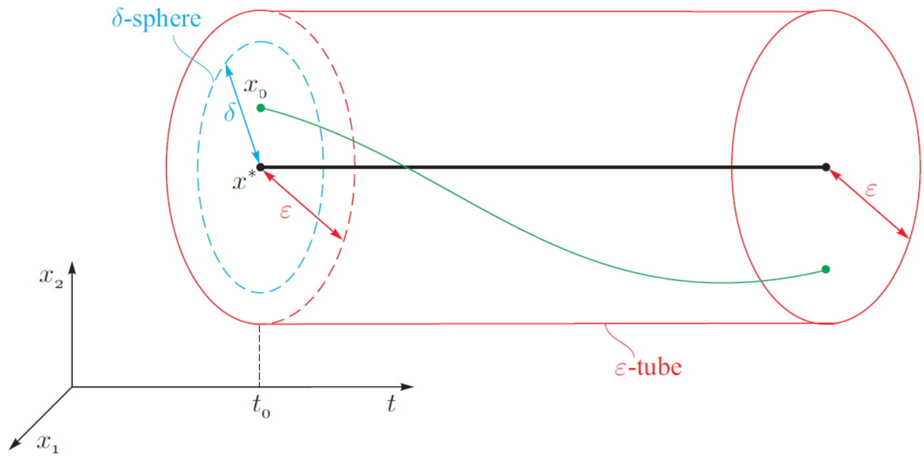

Figure 1: Lyapunov stable equilibrium.

Remark. The definition $\forall \delta > 0$ there exists an $\varepsilon > 0$ can be used to define the boundedness of the system. Note that boundedness and stability are *not* equivalent.

Definition 5 (Local/Global Attractiveness)**.** An equilibrium x^* is locally/globally attractive if

$$\exists \delta > 0 / \forall \delta > 0 : \quad \|x(t_0) - x^*\| < \delta \Longrightarrow \lim_{t \to \infty} \|x(t) - x^*\| = 0.$$

Definition 6. An equilibrium point which is stable and locally/globally attractive is locally/globally asymptotically stable.

Example 1. Consider the system $\dot{x} = -x$ with the equilibrium $x^* = 0$. We may prove that x^* is globally asymptotically stable as follows:

- stable: We know that $x(t) = x_0 e^{-t}$. Now, given $\varepsilon > 0$ can we find a $\delta > 0$ s.t. $\|x_0 - 0\| = \|x_0\| < \delta \Rightarrow \|x(t) - 0\| = \|x(t)\| < \varepsilon$? To ensure $\|x(t)\| < \varepsilon$, or equivalently $\|x_0 e^{-t}\| < \varepsilon$, it suffices to pick $\|x_0\| < \varepsilon$, since $\|e^{-t}\| \leq 1$. Hence, we may choose $\delta = \varepsilon$. As this holds for all $\varepsilon > 0$, the equilibrium point $x^* = 0$ is stable.

- globally attractive: for all $\delta > 0$ we have $\lim_{t\to\infty} x(t) = 0$, meaning that the equilibrium point is globally attractive.

Hence, the equilibrium point is globally asymptotically stable.

Let $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a continuous, strictly increasing function with $\alpha(0) = 0$ and $\lim_{q\to\infty} \alpha(q) = \infty$. Such functions are said to be of class $\mathcal{KL}$.

Example 2. The function $\alpha(q) = q^2$ is of class $\mathcal{KL}$ as $\alpha(0) = 0^2 = 0$ and $\lim_{q\to\infty} q^2 = \infty$.

Definition 7 ((Local) Positive Definiteness)**.** A function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is called:

1. locally positive definite (LPDF) if there exists a function $\alpha(\cdot)$ of class $\mathcal{KL}$ and a $h > 0$ such that $V(0) = 0$ and $V(x) \geq \alpha(\|x\|)$ for all $\|x\| < h$.

2. Positive definite (PDF) is 1 is true for all $h > 0$.

LDPF and PDF functions are also called candidate Lyapunov functions.

Remark. The following equivalences hold:

- $V(x)$ PDF if and only if $V(x) > 0 \,\forall x \neq 0, \lim_{\|x\|\to\infty} V(x) = \infty$.

- $V(x) = x^\top P x$ is PDF with symmetric P if and only if P is positive definite, i.e., $x^\top P x > 0 \,\forall x \neq 0$.

Example 3. The function $V(x) = 2x_1^2 + x_2^2$ is PDF as $V(0) = 0$ and $V(x) \geq \alpha(\|x\|)$ with $\alpha(\|x\|) = \|x\|^2 = x_1^2 + x_2^2$ for all $x \in \mathbb{R}^2$.

Example 4. The function $V(x) = x_1^2 + x_2^2/(1 + x_2^2)$ is LPDF but not PDF. We can verify that graphically in Figure 2.

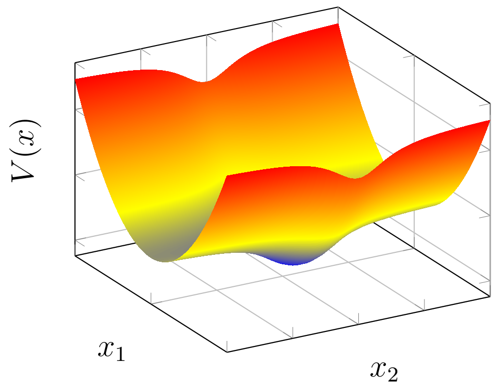

Figure 2: Plot of the Lyapunov function $V(x)$.

The total time derivative of $V(x)$ is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}^\top \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{\partial V}{\partial x}^\top f(x).\end{aligned} \tag{10.2}$$

Then, we can analyze the stability of the equilibrium point $x^* = 0$ as follows.

Theorem 1 (Basic Lyapunov Theorem)**.** Let $x^* = 0$ be an equilibrium of $\dot{x} = f(x)$:

1. If there exists $V(x)$ LPDF and $\dot{V}(x) \leq 0 \, \forall x$ with $\|x\| < h$ for some $h > 0$, $x^* = 0$ is stable.
2. If there exists $V(x)$ (L)PDF and $-\dot{V}(x)$ (L)PDF, $x^* = 0$ is locally/globally asymptotically stable.

Remark. The theorem provides *sufficient* conditions for stability, not *necessary* conditions. Hence, an equilibrium might be stable even if the Lyapunov theorem fails to prove stability.

Remark. There is no loss of generality in assuming $x^* = 0$. For a general equilibrium $x^* \neq 0$ we introduce the coordinate transformation $y = x - x^*$ and analyze the stability properties of $y^* = 0$.

10.4 Tips

No hints for this week's exercise session ☺.

10.5 Example

Your SpaghETH is now a powerhouse in the food truck industry. In a bid to further increase your revenues, you decide to start offering Tiramisù. As part of the procedure to prepare the famous Italian dessert, mascarpone cheese has to be thoroughly mixed with sugar. To save on electricity, you decide to do it manually with an old-fashioned ladle (oh boy, you're in for a wild ride). You opt to model this process as an actuated pendulum where you, unfortunately, are the actuator. Mascarpone is a firm cheese, and hence you realize that some sort of damping has to be taken into account.
Summarising, the dynamical model for your pendulum can be expressed in the form:

$$\ddot{\theta} + \dot{\theta} + \sin(\theta) = F\cos(t),$$

with $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$, and $\theta(t)$ describes the displacement with respect to the vertical axis.

1. Transform the system in the form $\dot{\bar{x}} = f(\bar{x}, t)$ with $\bar{x}(0) = \bar{x}_0$.

2. Transform the system in the form $\dot{\tilde{x}} = \tilde{f}(\tilde{x})$ with $\tilde{x}(0) = \tilde{x}_0$.

3. Set $F = 0$. Tranform the system in $\dot{x} = f(x)$ with $x(0) = x_0$ and find the equilibrium points.

4. Analyze the stability properties of the equilibrium point at the origin with the energy function as a Lyapunov function

$$E(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 + (1 - \cos(\theta)).$$

   Does the result match your physical intuition of the system?

5. Analyze the stability properties of the equilibrium point at the origin with an augmented energy as a Lyapunov function

$$E_{\text{aug}}(\theta, \dot{\theta}) = \frac{1}{4}\theta^2 + \frac{1}{2}\theta\dot{\theta} + E(\theta, \dot{\theta}).$$

6. Is the equilibrium at the origin globally asymptotically stable?

Solution.

1. Let $\bar{x}_1 = \theta$ and $\bar{x}_2 = \dot{\theta}$. The transformed system is
$$\begin{aligned}\dot{\bar{x}}_1 &= \bar{x}_2 \\ \dot{\bar{x}}_2 &= -\sin(\bar{x}_1) - \bar{x}_2 + F\cos(t),\end{aligned}$$
with $\bar{x}_0 = \begin{bmatrix}\theta_0 & \dot{\theta}_0\end{bmatrix}^\top$.

2. Let $\tilde{x}_1 = \theta, \tilde{x}_2 = \dot{\theta}$, and $x_3 = t$. The transformed system is
$$\begin{aligned}\dot{\tilde{x}}_1 &= \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= -\sin(\tilde{x}_1) - \tilde{x}_2 + F\cos(\tilde{x}_3) \\ \dot{\tilde{x}}_3 &= 1,\end{aligned}$$
with $\tilde{x}_0 = \begin{bmatrix}\theta_0 & \dot{\theta}_0 & 0\end{bmatrix}^\top$.

3. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. The transformed system is
$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin(x_1) - x_2,\end{aligned}$$
with $x_0 = \begin{bmatrix}\theta_0 & \dot{\theta}_0\end{bmatrix}^\top$.The equilibrium points are $x^* = \begin{bmatrix}k\pi & 0\end{bmatrix}^\top$ with $k \in \mathbb{Z}$.

4. Note that
$$V(x) = \frac{1}{2}x_2^2 + (1 - \cos(x_1))$$
is a valid candidate as it is locally positive definite. We have
$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \sin(x_1)\cdot x_2 + x_2 \cdot (-\sin(x_1) - x_2) = -x_2^2.$$
As $V(x) \leq 0$ (but not LPDF) we can conclude that the equilibrium point is stable.
Remark. The choice of the Lyapunov function $V(x)$, with its $1 - \cos(x_1)$ term, is motivated by the fact that we want $V(x)$ to be at least *locally* positive definite, as it's the case in the interval $(-2\pi, 2\pi)$.

5. Again, note that
$$V(x) = \frac{1}{4}x_1^2 + \frac{1}{2}x_1x_2 + \frac{1}{2}x_2^2 + (1 - \cos(x_1))$$
is a valid candidate as it is positive definite. We have
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) &= \left(\frac{1}{2}x_1 + \frac{1}{2}x_2 + \sin(x_1)\right)\cdot x_2 + \left(\frac{1}{2}x_1 + x_2\right)\cdot(-\sin(x_1) - x_2) \\ &= -\frac{1}{2}x_2^2 - \frac{1}{2}x_1\sin(x_1).\end{aligned}$$
As $x_1\sin(x_1) > 0$ for all $|x_1| < \pi, x_1 \neq 0$, we have $\dot{V}(x) < 0$ for all $\|x\| < h$ for some $h > 0$. Hence, the equilibrium point is locally asymptotically stable.

6. As the system has multiple equilibria, the equilibrium point at the origin cannot be globally asymptotically stable.