Exercise 4 - Hydraulic Systems

4.1 Hydraulic Systems

Hydraulic systems are, in general, described by the *Navier-Stokes equations* as you might have learned in fluid dynamics courses. In order to simplify the modeling of such systems, it is convenient to use simpler formulations. Typical elements which constitute hydraulic systems are:

- ducts,
- compressible nodes,
- valves (treated within the next few weeks).

4.1.1 Hydraulic Ducts

A sketch for a water duct is depicted in Figure 1. The quantities that we need to model a duct are the height difference h, the top and bottom pressures $p_1(t), p_2(t)$, the length of the duct l, the velocity of the water flowing into the duct $v(t)$, the density of the flowing fluid ρ, and the cross-sectional area of the duct A. In general, we are interested

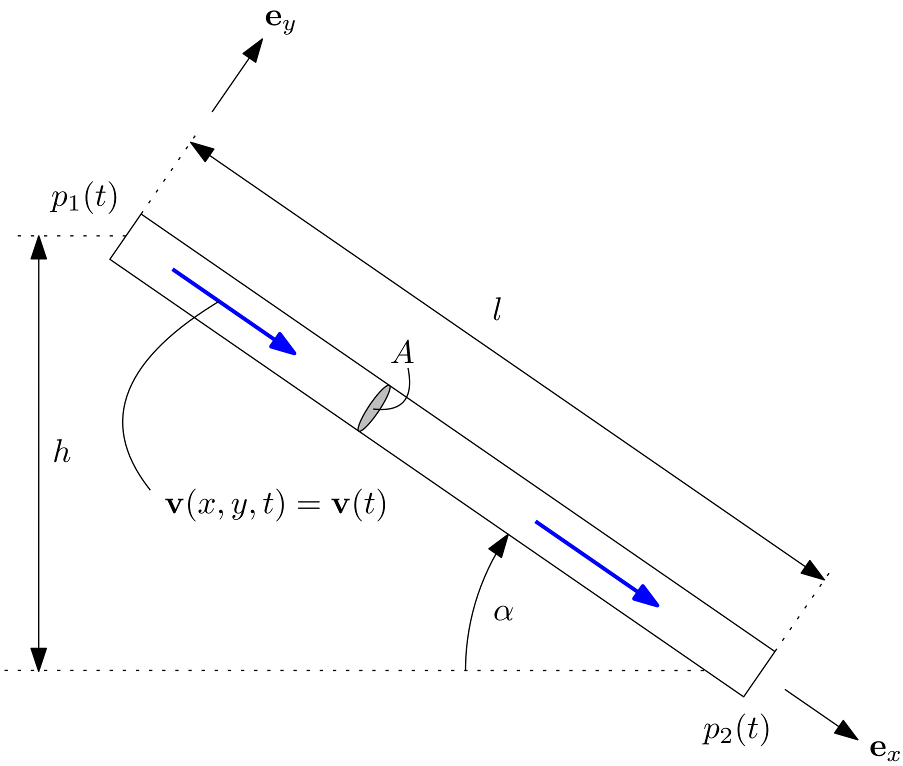

Figure 1: Sketch of a water duct.

in modeling the velocity of the fluid, i.e. finding an equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) = f(p_1(t), p_2(t), v(t), h, \rho, A, l). \tag{4.1}$$

A free body diagram of the fluid is shown in Figure 2.

This material is based on the HS17 teaching assistance taught by Nicolas Lanzetti and Gioele Zardini.
This document can be downloaded at `https://n.ethz.ch/~lnicolas`

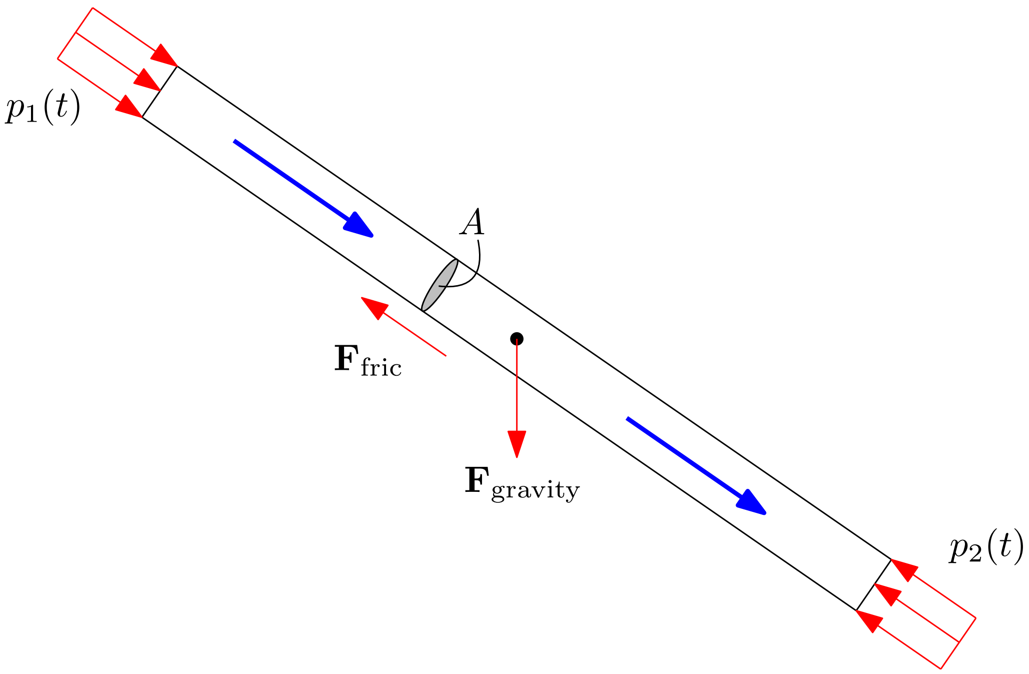

Figure 2: Free body diagram of the fluid.

Applying Newton's law to the chunk of water inside the duct gives

$$m \cdot \frac{\mathrm{d}v}{\mathrm{d}t} = F_{\text{pressure}} + F_{\text{gravity},x} - F_{\text{fric}}. \tag{4.2}$$

Under the assumption of a constant pressure distribution over the cross-sectional area A we may write

$$F_{\text{pressure}} = (p_1(t) \cdot A - p_2(t) \cdot A)$$

The mass of the fluid in the duct is given by

$$\rho \cdot V = \rho \cdot A \cdot l$$
$$\Rightarrow \mathrm{d}m = \rho \cdot A \cdot \mathrm{d}x.$$

To compute the contribution of gravity, we integrate over the whole duct

$$\begin{aligned}
\mathbf{F}_{\text{gravity}} &= \int_{\text{duct}} \mathbf{g} \cdot \mathrm{d}m \\
&= g \int_0^l \begin{bmatrix} \sin(\alpha) \\ -\cos(\alpha) \end{bmatrix} \cdot \rho \cdot A \cdot \mathrm{d}x \\
&= \rho \cdot g \cdot A \cdot l \cdot \begin{bmatrix} \sin(\alpha) \\ -\cos(\alpha) \end{bmatrix}
\end{aligned}$$

Using the facts that the elevation angle α satisfies

$$\sin(\alpha) = \frac{h}{l},$$

we may write

$$F_{\text{gravity},x} = h \cdot \rho \cdot g \cdot A.$$

The friction force depends on the shape factor $\frac{l}{d}$ and reads

$$F_{\text{fric}} = \frac{1}{2} \cdot \rho \cdot v^2 \cdot \text{sign}(v) \cdot \lambda \cdot \frac{Al}{d} \tag{4.3}$$

where λ is a constant coefficient and d is the diameter of the pipe. Plugging the found results in Equation (4.2) one gets the conservation law along the longitudinal axis of the duct

$$m\frac{\mathrm{d}}{\mathrm{d}t}v(t) = \rho \cdot l \cdot A \cdot \frac{\mathrm{d}}{\mathrm{d}t}v(t) = A \cdot (p_1(t) - p_2(t)) + A \cdot \rho \cdot g \cdot h - F_{\text{fric}}(t). \tag{4.4}$$

4.1.2 Compressibility of Ducts

The **compressibility** is the property of ducts (and possibly other fluid containers) to deform under the effect of an applied pressure. Mathematically, it is defined as

$$\sigma_0 = \frac{1}{V_0}\frac{\partial V}{\partial p}, \tag{4.5}$$

where V [m^3] is the volume, p [Pa] is the pressure, and σ_0 [Pa^{-1}] the compressibility.

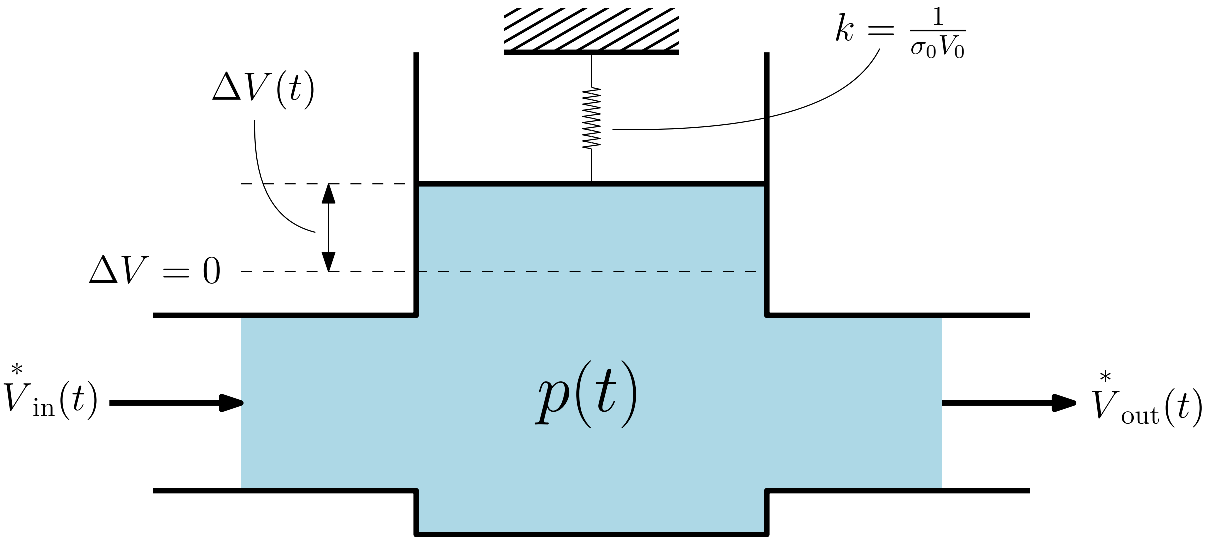

Figure 3: Sketch of the hydraulic compressibility

In general, since the fluid entering the element is not equal to the one exiting, we have a time-varying volume

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = \overset{*}{V}_{\text{in}}(t) - \overset{*}{V}_{\text{out}}(t). \tag{4.6}$$

Changes in volume cause a direct increase in the acting pressure according to

$$p(t) = \underbrace{\frac{1}{\sigma_0} \cdot \frac{\Delta V(t)}{V_0}}_{\Delta p_{\text{compress}}} + p_0, \qquad \Delta V(t) = V(t) - V_0, \tag{4.7}$$

where p_0 and V_0 denote the static (unloaded) pressure and volume of the elastic element respectively.

4.1.3 Compressibility of Fluids

Analogously, we define compressibility of matter (fluids in particular) as

$$\sigma_0 = -\frac{1}{V_0}\frac{\partial V}{\partial p}, \tag{4.8}$$

where V [m^3] is the volume, p [Pa] is the pressure, and σ_0 [Pa^{-1}] the compressibility. Note that the only difference with the above is a minus sign in the definition, that is

$$p(t) = -\frac{1}{\sigma_0}\frac{\Delta V(t)}{V_0} + p_0 \qquad \Delta V(t) = V(t) - V_0. \tag{4.9}$$

4.2 Tips

Formulate the mass balance for the water tank as

$$\frac{\mathrm{d}}{\mathrm{d}t} m_W = \rho \cdot A_W \cdot \frac{\mathrm{d}}{\mathrm{d}t} \bar{h}_W = \dot{m}_{\text{in}} - \dot{m}_{\text{out}}$$

with a fictitious height $\bar{h}_W$. Then, you may find the true height h_W with

$$h_W = \bar{h}_W + \lambda_W \cdot \operatorname{sign}\left(\frac{\mathrm{d}}{\mathrm{d}t} h_W\right) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t} h_W\right)^2.$$

Moreover, the mass flow of a incompressible fluid through a valve with opening area A can be modeled as

$$\dot{m}(t) = c_{\text{d}} \cdot A \cdot \sqrt{2\rho \cdot (p_{\text{before}} - p_{\text{after}})},$$

where c_{d} is the so-called discharge coefficient.

4.3 Example

Your SpaghETH is growing every week more and although no particular production issues occur you are concerned about ecology. Since each tank of pasta you cook needs water and a correct salt seasoning for it to taste that delicious, you need a lot of salt and water, which are often wasted. For this reason, you open a research branch in your startup which decides to design a duct-hydraulic system to counteract the waste of water and salt. The tank where the pasta cooks is connected to a duct (diameter d_T), the *Tunnel*. Before the water enters a second duct (diameter d_T), the *Seasoner*, a pump increases its pressure by Δp. Note that only compressibility effects of the *Tunnel* should be taken into account. The pressure at the water's surface p_∞ is assumed to be known. Assume a circular tunnel, whose area reads $A_\mathrm{T} = \pi d_\mathrm{T}^2/4$. The area of the water tank is A_W (with $A_\mathrm{W} \gg A_\mathrm{T}$). A sketch of the system with the relative parameters is shown in Figure 4.

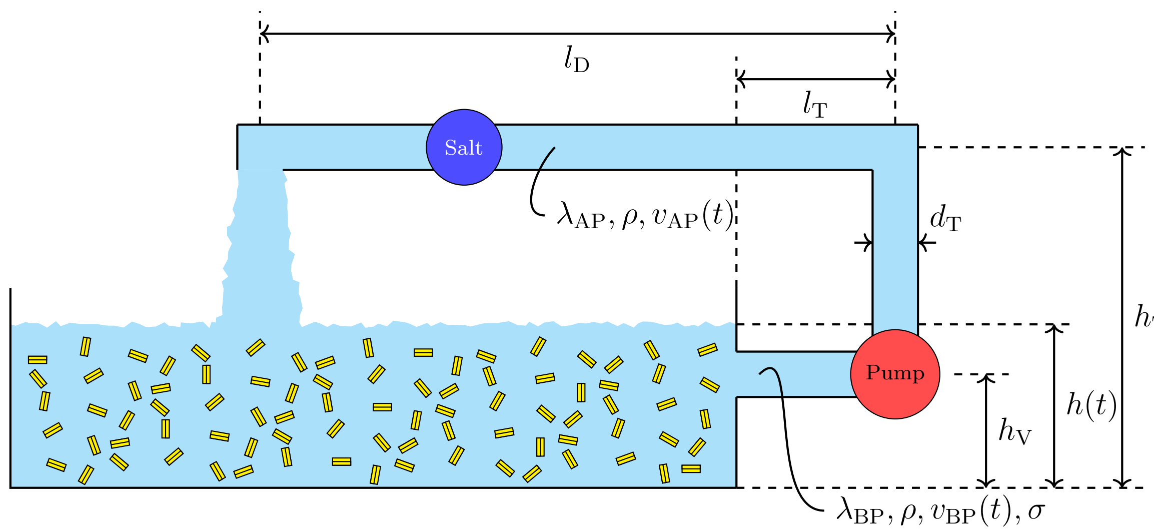

Figure 4: Sketch of the system.

1. List all the reservoirs and the relative state variables.
2. Find the pressure $p_1(t)$ at the beginning of the *Tunnel* as a function of the velocity in the *Tunnel* $v_\mathrm{BP}(t)$.
3. Formulate the differential equation for $v_\mathrm{BP}(t)$ as function of the pressure right before the pump $p_2(t)$.
4. Exploiting the compressibility of the *Tunnel*, find the pressure $p_2(t)$ explicitly.
5. Formulate the differential equation for $v_\mathrm{AP}(t)$.
6. Formulate the differential equation for $h(t)$.

Solution.

1. The reservoirs of the system and their relative state variables are:
   - The *Tunnel* kinetic energy. The state variable of this reservoir is the velocity of the water in the *Tunnel* $v_{\mathrm{BP}}(t)$.
   - The *Seasoner* kinetic energy. The state variable of this reservoir is the velocity of the water in the *Tunnel* $v_{\mathrm{AP}}(t)$.
   - The water mass in the tank. The state variable of this reservoir is the water's height $h(t)$.
   - The compressibility of the *Tunnel*. The state variable of this reservoir is the water volume $V(t)$.

2. **Bernoulli:**
   First, we need to use Bernoulli's law from the water's surface to the beginning of the *Tunnel*, in order to find the local pressure. This reads

$$\begin{aligned} 0^2 + \frac{p_\infty}{\rho} + g \cdot h(t) &= \frac{v_{\mathrm{BP}}(t)^2}{2} + \frac{p_1(t)}{\rho} + g \cdot h_{\mathrm{V}} \\ p_1(t) &= p_\infty + g \cdot \rho \cdot (h(t) - h_{\mathrm{V}}) - \rho \cdot \frac{v_{\mathrm{BP}}(t)^2}{2}. \end{aligned} \tag{4.10}$$

   Here, we are making the assumption that the area of the tank is much larger than that of the tunnel, thus $\dot{h}(t) \approx 0$.

3. **Tunnel:**
   The impulse equation for the *Tunnel* reads

$$\rho \cdot l_{\mathrm{T}} \cdot A_{\mathrm{T}} \cdot \frac{\mathrm{d}}{\mathrm{d}t} v_{\mathrm{BP}}(t) = A_{\mathrm{T}} \cdot (p_1(t) - p_2(t)) - F_{\mathrm{fric}}(t), \tag{4.11}$$

   where $F_{\mathrm{fric}}(t)$ is the friction force, which reads

$$F_{\mathrm{fric}}(t) = A_{\mathrm{T}} \cdot \lambda_{\mathrm{BP}} \cdot \frac{l_{\mathrm{T}} \cdot \rho}{2 \cdot d_{\mathrm{T}}} \cdot \mathrm{sign}(v_{\mathrm{BP}}(t)) \cdot v_{\mathrm{BP}}(t)^2. \tag{4.12}$$

   The pressure $p_2(t)$ is obtained from the compressibility of the *Tunnel*.

4. **Compressibility:**
   As learned in the lecture, the pressure before the valve $p_2(t)$ is computed as

$$\begin{aligned} p_2(t) &= \frac{1}{\sigma} \cdot \frac{\Delta V(t)}{V_0} + p_{\mathrm{stat}} \\ &= \frac{1}{\sigma} \cdot \frac{V(t) - V_0}{V_0} + \rho \cdot g \cdot (h(t) - h_{\mathrm{V}}) + p_\infty, \end{aligned} \tag{4.13}$$

   where

$$V_0 = \frac{l_{\mathrm{T}} \cdot \pi \cdot d_{\mathrm{T}}^2}{4}. \tag{4.14}$$

   The volume balance reads

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &= V_{\mathrm{in}}^* - V_{\mathrm{out}}^* \\ &= A_{\mathrm{T}} \cdot (v_{\mathrm{BP}}(t) - v_{\mathrm{AP}}(t)). \end{aligned} \tag{4.15}$$

5. **Seasoner:**
   The impulse equation for the *Seasoner* reads

$$\rho\cdot(l_{\mathrm{D}}+h_{\mathrm{T}}-h_{\mathrm{V}})\cdot A_{\mathrm{T}}\cdot\frac{\mathrm{d}}{\mathrm{d}t}v_{\mathrm{AP}}(t)=A_{\mathrm{T}}\cdot(p_2(t)+\Delta p-p_{\infty})+A_{\mathrm{T}}\cdot\rho\cdot g\cdot(h_{\mathrm{V}}-h_{\mathrm{T}})-F_{\mathrm{fric}}(t), \tag{4.16}$$

   where $F_{\mathrm{fric}}(t)$ is the friction force, which reads

$$F_{\mathrm{fric}}(t)=A_{\mathrm{T}}\cdot\lambda_{\mathrm{AH}}\cdot\frac{(l_{\mathrm{D}}+h_{\mathrm{T}}-h_{\mathrm{V}})\cdot\rho}{2\cdot d_{\mathrm{T}}}\cdot\mathrm{sign}(v_{\mathrm{AP}}(t))\cdot v_{\mathrm{AP}}(t)^2. \tag{4.17}$$

6. **Water Tank:**
   The water tank stores water's mass. Its state variable is the height of the water tank $h(t)$. The mass balance reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}m(t)&=\rho\cdot A_{\mathrm{W}}\cdot\frac{\mathrm{d}h(t)}{\mathrm{d}t}\\&=\overset{*}{m}_{\mathrm{in}}-\overset{*}{m}_{\mathrm{out}}\\&=\rho\cdot A_{\mathrm{T}}\cdot(v_{\mathrm{AP}}(t)-v_{\mathrm{BP}}(t)).\end{aligned} \tag{4.18}$$

   This leads to a relation for the change of the water height

$$\frac{\mathrm{d}}{\mathrm{d}t}h(t)=\frac{A_{\mathrm{T}}}{A_{\mathrm{W}}}\cdot\left(v_{\mathrm{AP}}(t)-v_{\mathrm{BP}}(t)\right). \tag{4.19}$$