Exercise 1 - Introduction

1.1 Aim of the Course

The aim of this course is to come up with *mathematical representations* or *models* for given systems. We distinguish, in particular, between three purposes:

- Analysis and understanding,
- simulation,
- control.

1.2 Preliminary Definitions

We start by stating some fundamental definitions, which will often be used throughout the course.

Definition 1. A **dynamic system** is a non static system which evolves with respect to time. The time evolution can be caused by

- input signals,
- external perturbations, and/or
- its natural evolution.

Examples of time evolutions are multiple:

- Trajectory changes (position, velocity, and acceleration),
- physical properties changes (temperature, pressure, volume, etc.).

Mathematical models of dynamic systems can be subdivided in two major classes:

- parametric models,
- non-parametric models.

Definition 2. A **parametric model** describes a system through its parameters.

Example 1. Mechanical systems are described with the general differential equation

$$m\ddot{y}(t) + d\dot{y}(t) + ky(t) = F(t),$$

where the parameters are: mass m, viscous damping coefficient d, and spring constant k.

Definition 3. A **non-parametric model** describes the system through the knowledge of its *actual* response to an actual given input.

Example 2. Knowing the impulse response of a damped mechanical oscillator.

In this course, we will only consider parametric models. We can further distinguish between three different types of models:

- **Black-box models**: derived from experiments only.
- **Grey-box models**: starting from a model, one uses experiments to identify model parameters and validate the model.
- **White-box models**: no experiments needed, since one knows each part of the system.

This material is based on the HS17 teaching assistance taught by Nicolas Lanzetti and Gioele Zardini.
This document can be downloaded at `https://n.ethz.ch/~lnicolas`

1.3 How to Build a Model

When modeling a system, two big groups of actors are important to be defined:

Definition 4. The **reservoirs** are accumulative elements. Examples of reservoirs are various energies, mass or information. Moreover, two important aspects should be underlined:

- Only systems including one or more reservoirs exhibit dynamic behaviour and
- for each reservoir an associated *level* variable can be defined.

Remark. The choice of the level variable is not always unique.

Definition 5. Flows describe the flowing of heat, mass, etc. between reservoirs.

In order to be able to model a system, we propose the following methodology.

Reservoir-based Approach

For a given system:

(I) Define the system boundaries:

- Inputs: signals *entering* the system boundaries.
- Outputs: signals *exiting* the system boundaries.

(II) Identify the relevant reservoirs and the corresponding level variables.

(III) Formulate the differential equations which decribe the relevant reservoirs. These are usually conservation laws and can be generally expressed as:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{reservoir content}) = \sum \text{inflows} - \sum \text{outflows}. \tag{1.1}$$

(IV) Formulate the algebraic relations which express the flows between the reservoirs. These should be expressed as a function of the level variables, i.e.,

$$\text{flow} = f(\text{level variables}, \text{parameters}, \ldots). \tag{1.2}$$

(V) Solve implicit algebraic equations and simplify the resulting mathematical relations as much as possible.

(VI) Identify the unknown system parameters using experiments.

(VII) Validate the model with experiments that differ from the ones used to identify the parameters.

Remark. With relevant dynamics one refers to the variables whose time constant is of the same of magnitude as the one to the sytem. Too quick dynamics translate to algebraic relations, whereas too slow dynamics lead to constant variables.

Causality Diagram

A causality diagram is a graphical representation of the system's equations. Here, we distinguish between algebraic and dynamic blocks, as shown in Figure 1. A dynamic block is characterized by a differential equation, while an algebraic block is defined by a static equation. The blocks are connected through arrows, which represent variables.

Figure 1: Block types.

Remark. In general, there are multiple ways to draw the causality diagram of a system.

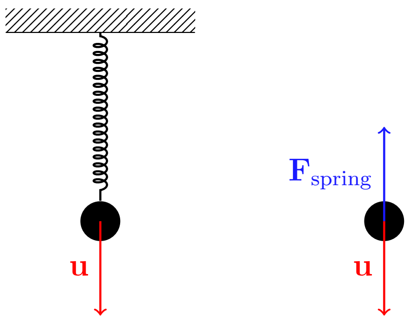
Example 3. For a general, forced, mass-spring mechanical system with mass m and spring constant k, one can write the Newton's law as

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = u - F_{\text{spring}},$$

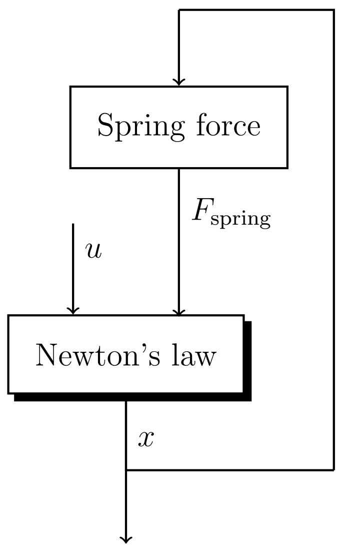
where

$$F_{\text{spring}} = kx$$

is the spring force. In this case, one can observe two types of relations: Newton's law contains derivatives in time and hence dynamics. The spring force is an algebraic relation ($y = f(x)$). Hence, the causality diagram can be represented as shown in Figure 2b.



(a) Sketch of the system.



(b) Causality diagram of the system.

Figure 2: Mechanical oscillator.

1.4 Tips

Exercise 1: No tips. ☺

Exercise 2: You may find the Bernoulli equation useful:

$$\frac{1}{2}\rho v_1^2 + p_1 + \rho g h_1 = \frac{1}{2}\rho v_2^2 + p_2 + \rho g h_2.$$

Why is it possible to apply it?

How can you simplify it with the given values in Table 1?

Exercise 3: For Problem e) recall that at equilibrium $\frac{\mathrm{d}}{\mathrm{d}t}(\cdot) = 0$.

1.5 Example

In recent years, the market of food trucks has seen a tremendous increase. Willing to get into the market, you decide to found the company SpaghETH. Your focus is to serve fresh pasta with different sauces everyday. In order not to run into delays during your activity, you aim to have hot water right after parking and therefore you decide to warm up the water while driving. To find the optimal way to that goal, you start by formulating a model of the system.

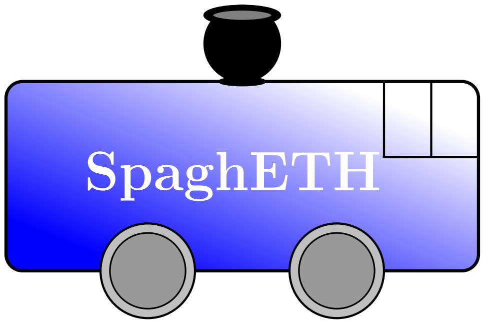


Figure 3: Sketch of the system.

The truck is modeled as a point mass of mass m. The propulsive power acting on the truck is given by

$$P_{\mathrm{p}}(\theta) = P_{\max} \cdot (1 - \exp(-c_1\theta)),$$

where $\theta(t)$ is the (normalized) angular position of the pedal and c_1 and $P_{\max}$ are known constants. The truck is also subject to the aerodynamic drag force

$$F_{\mathrm{drag}} = \frac{1}{2} c_2 v^2,$$

where c_2 is a known constant and v is the speed of the truck. Clearly, your truck is equipped with a pot of known area A and volume V. Your cooker allows you to set the heat flow given to the water. The heat transfer coefficient α between the water surface and the air is a known function of the velocity of the truck. The ambient temperature T_∞ as well as the specific heat c and the density ρ of water are known. You may assume that no water evaporates. Furthermore, assume that the mass of the truck is much larger than the mass of the driver and of the equipment. Your truck is equipped with one sensor measuring the velocity and a temperature sensor.

1. Determine the inputs and the outputs of the system.
2. List the reservoir(s) and the corresponding level variable(s).
3. Draw a causality diagram of the system.
4. Formulate the differential/algebraic equations needed to describe the system.
5. Is the system linear or nonlinear? Explain.

Solution.

1. The inputs are the position of the pedal θ and the heat transfer to the water $\dot{Q}$. The outputs are the temperature of the water in the pot and the velocity of the truck.

2. The system has two reservoirs:

   - the kinetic energy of the truck E_{kin}, whose level variable is the velocity v of the truck;
   - the thermal energy of the water in the pot U, whose level variable is the temperature of the water in the pot.

   Note that, since no vapour leaves the water, the mass of the water in the system is not a reservoir.

3. The causality diagram is shown in Figure 4.

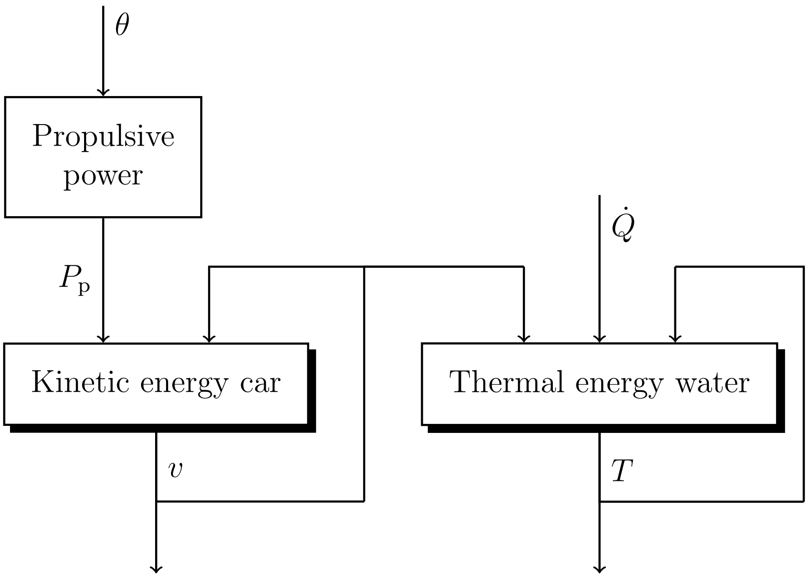


Figure 4: Causality diagram of the system.

4. The algebraic equation that relates the propulsive power and the position of the pedal reads

$$P_{\text{p}}(\theta) = P_{\max} \cdot (1 - \exp(-c_1\theta))\,.$$

The differential equation for the kinetic energy of the car

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{\text{kin}} &= P_{\text{in}} - P_{\text{out}} \\ &= P_{\text{p}} - F_{\text{drag}} \cdot v\end{aligned}$$

leads to

$$mv \cdot \frac{\mathrm{d}}{\mathrm{d}t}v = P_{\text{p}} - \frac{1}{2}c_2 v^3,$$

where we used $\frac{\mathrm{d}}{\mathrm{d}t}v^2 = 2v\frac{\mathrm{d}}{\mathrm{d}t}v$. The heat balance of the pot

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}U &= P_{\text{in}} - P_{\text{out}} \\ &= \dot{Q} - \dot{Q}_{\text{loss}}\end{aligned}$$

gives

$$\rho V c \cdot \frac{\mathrm{d}}{\mathrm{d}t}T = \dot{Q} - \alpha(v)A \cdot (T - T_\infty).$$

5. The system is nonlinear. Examples of nonlinearities are the relation between the pedal angular position and the propulsive power and the dynamics of the car.