A MIMO Nyquist Stability Test Example

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The MIMO Nyquist criterion is used to assess the stability of a MIMO system under closed-loop negative feedback. The MIMO system must be square and the feedback is $-I$. In other words, unity gain negative feedback on each input-output channel.

The criterion is very similar to the more commonly known SISO Nyquist stability test. However some care is needed in defining the boundary of the right-half plane, particularly when it must be perturbed to avoid poles on the $j\omega$-axis.

The example given here illustrates that $j\omega$-axis poles should be counted as unstable when one is counting the correct number of encirclements to determine closed-loop stability. This example is similar that mentioned in a paper on MIMO Nyquist stability[1]. It is rather simple and really illustrates a case where the feedback has no affect on one of the unstable parts of the system.

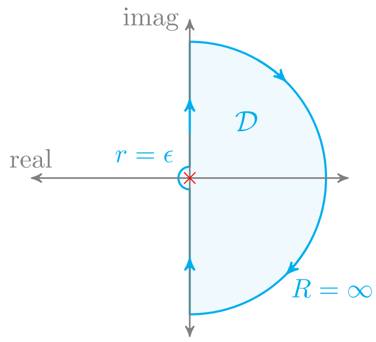
1 MIMO Nyquist stability criterion

First we recall the MIMO Nyquist stability criterion. We have an open-loop MIMO transfer function, $L(s)$, which is typically a loop transfer function. The criterion determines whether or not the closed-loop unity negative gain feedback transfer function, $S(s) = (I + L(s))^{-1}$ is stable. The criterion uses Cauchy's principle of the argument to determine the number of poles of $S(s)$ in the right-half plane (RHP).

To begin we define a closed contour, denoted here by $\bar{\mathcal{D}}$. The interior of the contour is denoted by $\mathcal{D}$, and the criterion will determine how many closed-loop poles lie in $\mathcal{D}$. To make this applicable to testing stability we will chose $\mathcal{D}$ to be (approximately) the entire RHP. We will see that there are some subtleties in how this should be done.

We require that $L(s)$ is finite for all s on the contour boundary, $\bar{\mathcal{D}}$. This means that $\bar{\mathcal{D}}$ must not pass through any poles of $L(s)$. A typical definition of the region $\mathcal{D}$ is shown below.

[1] C.A. Desoer and Y.-T. Wang, "On the generalized Nyquist stability criterion," *IEEE Trans. Automatic Control,* vol. 25, no. 2, pp. 187–196, 1980.



This illustrates a choice of $\mathcal{D}$ for an open-loop transfer function, $L(s)$, with a pole at $s = 0$. To avoid the contour, $\bar{\mathcal{D}}$, going through this pole, $\bar{\mathcal{D}}$ makes a radius r indentation into the left-half plane (LHP) to go around the pole. Typically r is chosen to be very small, $r = \epsilon$, or we consider the limit in which $r \longrightarrow 0$. A similar procedure is used for any other poles on the $j\omega$ axis.

To capture the entire RHP we define the boundary on the right of $\mathcal{D}$ by an arc of radius R and consider the limit as $R \longrightarrow \infty$. This part of the definition of $\bar{\mathcal{D}}$ is not critical as for real-rational transfer functions $L(s)$ is equal to a constant (typically zero) for all s on the radius R part of the boundary of $\mathcal{D}$.

Given the definition of $\mathcal{D}$ above, we can state the stability theorem.

Theorem 1 *The closed-loop system,*

$$S(s) = (I + L(s))^{-1}$$

is exponentially stable, if and only if,

i) $\det(I + L(s)) \neq 0$ *for all* s *on* $\bar{\mathcal{D}}$*; and*

ii) The contour of

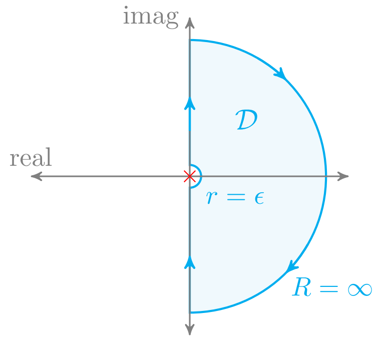
$$\det(I + L(s)),$$

where s *traverses* $\bar{\mathcal{D}}$ *in a clock-wise direction, encircles the origin* $p_{\mathcal{D}}$ *times in a counter-clockwise direction, where* $p_{\mathcal{D}}$ *is the number of poles of* $L(s)$ *in* $\mathcal{D}$*.*

Condition *i)* is a *well-posedness* condition. If it is not satisfied the equations for calculating the input and output signals for $S(s)$ do not have a unique solution. In such cases the feedback interconnection is not well defined.

The most significant difference between the MIMO condition in Theorem 1 and that for the SISO case is the use of the determinant in the calculation of the contour. Note that in the SISO case the determinant of $(I + L(s))$ is simply $1 + L(s)$, and the number of encirclements of the origin by $1 + L(s)$ is equal to the number of encirclements of -1 by $L(s)$. The tests are the same in this case.

It is common in SISO Nyquist stability analysis to define the contour as deviating into the RHP to avoid poles on the $j\omega$-axis. This case is illustrated below. The pole at $s = 0$ is now considered to be outside of $\mathcal{D}$, which reduces the number of encirclements required.



We will look an example using both of these region definitions and see that avoiding the poles on the $j\omega$-axis by moving the contour into the RHP (as in the second illustration) may give an erroneous result.

2 Example loop transfer function

A simple two-input, two-output loop is considered. In this example, the cause of the difficulties with respect to the definition of $\mathcal{D}$ is easy to see. In more complex systems of higher input-output dimensions this may not be the case.

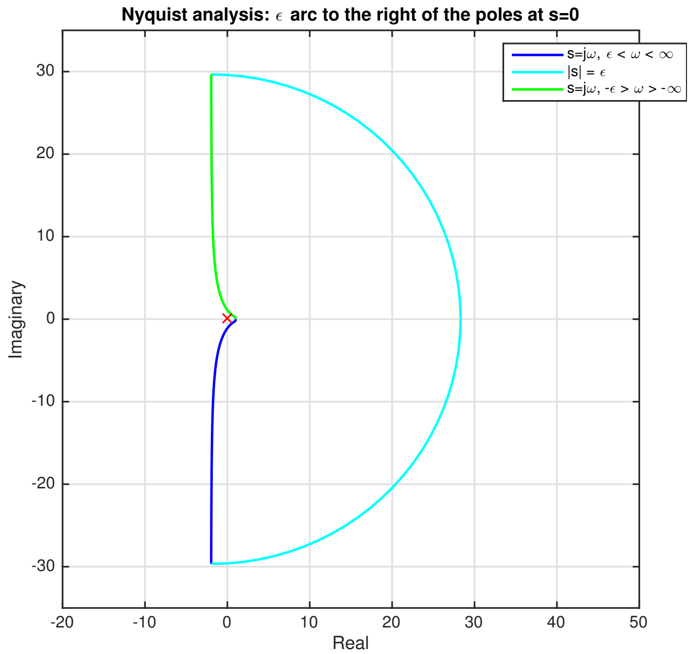
The loop transfer function to be considered is,

$$L(s) = \begin{bmatrix} \frac{2}{s(s+1)} & \frac{1}{s^2} \\ 0 & \frac{1}{(s+2)} \end{bmatrix}.$$

This system has a pole at $s = -2$, a pole at $s = -1$ and two poles at $s = 0$.

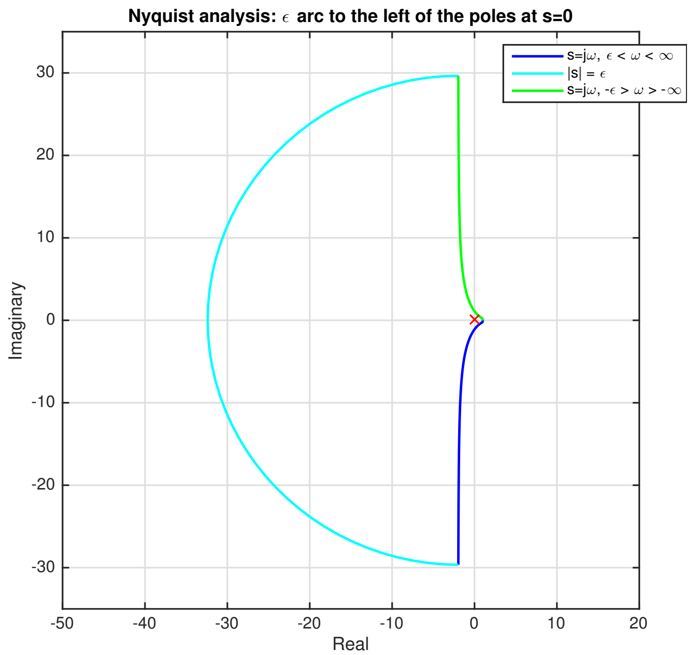
3 MIMO Nyquist analysis

First consider the typical SISO definition of $\mathcal{D}$; the ϵ arc goes to the right of any poles on the $j\omega$-axis. In this case $L(s)$ has no poles in $\mathcal{D}$. For closed-loop stability we therefore require that $\det(I + L(s))$, evaluated clockwise around $\bar{\mathcal{D}}$, has no counter-clockwise encirclements of the origin. The result of this evaluation is shown below.



There are no counter-clockwise encirclements of the origin, implying that the closed-loop transfer function is exponentially stable.

Now consider the definition of $\mathcal{D}$ where the ϵ arcs are defined to go to the left of any poles on the $j\omega$-axis. In this case, the two poles of $L(s)$ at $s = 0$ are now within $\mathcal{D}$. Therefore closed-loop stability will require two counter-clockwise encirclements of the origin. The evaluation of $\det(I + L(s))$ along this particular $\bar{\mathcal{D}}$ is shown below.



This shows only one counter-clockwise encirclement of the origin meaning that the closed-loop system is not exponentially stable. The fact that it is one encirclement less than the required number implies that there is one closed-loop pole within $\mathcal{D}$.

These two analyses are clearly in conflict.

4 Simulation illustration

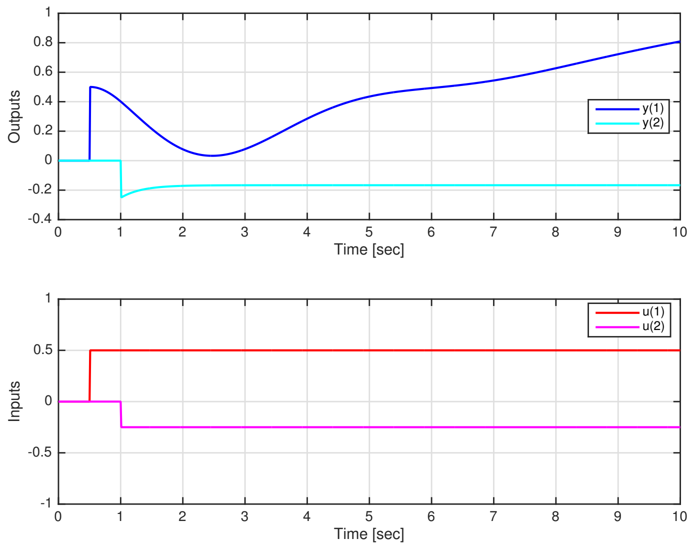
The stability of the closed-loop system is illustrated by a simulation. Define closed-loop inputs and outputs via,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = S(s) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The outputs, y_1 and y_2 are calculated for the inputs,

$$\begin{aligned} u_1 &= 0.5 \,\text{step}(t - 0.5) \\ u_2 &= -0.25 \,\text{step}(t - 1.0). \end{aligned}$$

The results are illustrated below.



The output $y_1(t)$ grows without bound. The closed-loop system is not bounded-input, bounded-output (BIBO) stable.

The prediction of instability given by the analysis with the ϵ arcs to the left of the singularities is clearly correct.

5 Transfer function calculation

The result can also be confirmed analytically. We have,

$$L(s) = \begin{bmatrix} \dfrac{2}{s(s+1)} & \dfrac{1}{s^2} \\ 0 & \dfrac{1}{(s+2)} \end{bmatrix},$$

and so,

$$\begin{aligned}
(I + L(s))^{-1} &= \begin{bmatrix} 1 + \dfrac{2}{s(s+1)} & \dfrac{1}{s^2} \\ 0 & 1 + \dfrac{1}{(s+2)} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \dfrac{(s^2+s+2)}{s(s+1)} & \dfrac{1}{s^2} \\ 0 & \dfrac{(s+3)}{(s+2)} \end{bmatrix}^{-1} \\
&= \frac{1}{\left(\dfrac{(s^2+s+2)}{s(s+1)} \dfrac{(s+3)}{(s+2)} - 0 \dfrac{-1}{s^2} \right)} \times \begin{bmatrix} \dfrac{(s+3)}{(s+2)} & \dfrac{-1}{s^2} \\ 0 & \dfrac{(s^2+s+2)}{s(s+1)} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{s(s+1)}{(s^2+s+2)} & \dfrac{-(s+1)(s+2)}{s(s^2+s+2)(s+3)} \\ 0 & \dfrac{(s+2)}{(s+3)} \end{bmatrix}.
\end{aligned}$$

The diagonal elements of the closed-loop system have an oscillatory pole pair at $s = -0.5 \pm j\sqrt{7/4}$ and pole at $s = -3$. However the $S_{1,2}(s)$ term has these three poles as well as an unstable pole at $s = 0$.

6 Remarks

The source of the problems in analysing this system comes from the fact the open-loop system, $L(s)$, has two unstable poles at $s = 0$, but one of them is not affected by the feedback. It remains at $s = 0$ even in closed-loop. This phenomenon doesn't happen in the SISO case—all of the poles are moved by feedback.

The problem is also evident in the determinant calculation. The $1/s^2$ term in $L_{1,2}(s)$ multiplies the zero term in $L_{2,1}(s)$ in the determinant calculation. One of the integrators also appears in the $L_{1,1}(s)$ term and is stabilised by the feedback. The other is not.

Although it is fairly simple to see what happens in this case, more complex situation may not be as obvious.

Defining $\mathcal{D}$ to include any poles of $L(s)$ on the $j\omega$-axis avoids this difficulty and gives the correct stability criterion.