

Control Systems 2

Lecture 8: MIMO stability and stabilisation

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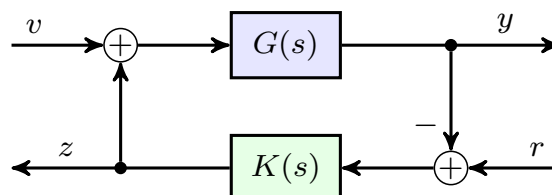
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8.1

Internal stability

Definition

A system is internally stable if for all initial conditions, and all bounded signals injected at any place in the system, all states remain bounded for all future time.



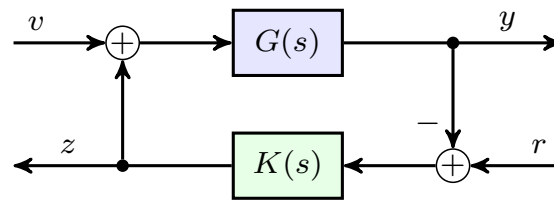
$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

Are all four transfer functions stable?

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Internal stability



Closed-loop input-output relationship:

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

Are all four transfer functions stable?

Example:

$$G(s) = \frac{s-1}{s+1} \quad K(s) = \frac{k(s+1)}{s(s-1)}$$

Work out $S(s)$, $T(s)$, and all four transfer functions, $N_{ij}(s)$.

MIMO concepts: transfer function matrices

$$y(s) = \begin{bmatrix} y_1(s) \\ \vdots \\ y_{n_y}(s) \end{bmatrix} = G(s)u(s) = \begin{bmatrix} G_{11}(s) & \dots & G_{1n_u}(s) \\ \vdots & & \vdots \\ G_{n_y1}(s) & \dots & G_{n_y n_u}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ \vdots \\ u_{n_u}(s) \end{bmatrix}$$

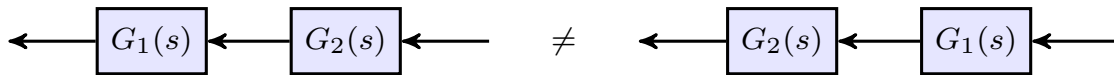
$$G(s) = \begin{bmatrix} G_{11}(s) & \dots & G_{1n_u}(s) \\ \vdots & & \vdots \\ G_{n_y1}(s) & \dots & G_{n_y n_u}(s) \end{bmatrix} = \begin{bmatrix} \frac{b_{11}(s)}{a_{11}(s)} & \dots & \frac{b_{1n_u}(s)}{a_{1n_u}(s)} \\ \vdots & & \vdots \\ \frac{b_{n_y1}(s)}{a_{n_y1}(s)} & \dots & \frac{b_{n_y n_u}(s)}{a_{n_y n_u}(s)} \end{bmatrix}$$

$$= C(sI - A)^{-1}B + D = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right],$$

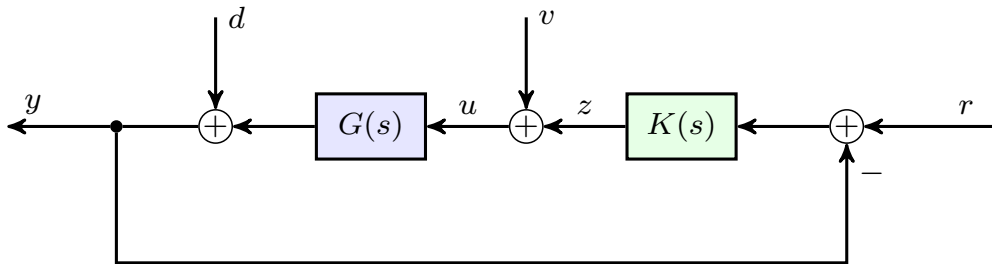
$$\text{with } A \in \mathcal{R}^{n \times n}, B \in \mathcal{R}^{n \times n_u}, C \in \mathcal{R}^{n_y \times n}, D \in \mathcal{R}^{n_y \times n_u}.$$

MIMO block diagrams

Non-commutative

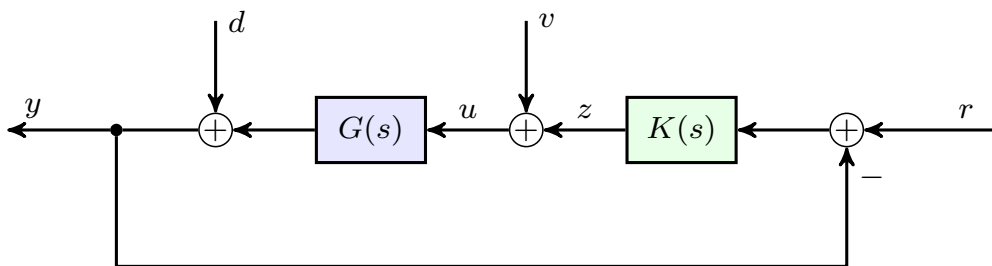


“Push-through” rule



$$GK(I + GK)^{-1} = G(I + KG)^{-1}K = (I + GK)^{-1}GK$$

MIMO sensitivity and complementary sensitivity functions

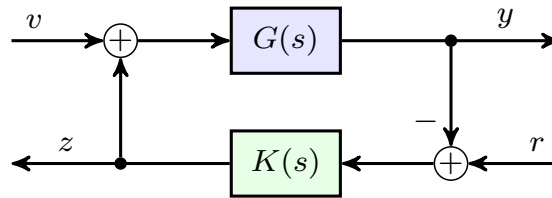


$$y = \underbrace{(I + GK)^{-1}GK}_{T_o} r + (I + GK)^{-1}Gv + \underbrace{(I + GK)^{-1}}_{S_o} d$$

$$u = (I + KG)^{-1}Kr + \underbrace{(I + KG)^{-1}}_{S_i} v - (I + KG)^{-1}Kd$$

$$z = (I + KG)^{-1}Kr - \underbrace{(I + KG)^{-1}KG}_{T_i} v - (I + KG)^{-1}Kd$$

Internal stability



$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix} = \begin{bmatrix} S_o(s)G(s) & T_o(s) \\ -T_i(s) & S_i(s)K(s) \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

$$(S_o = (I + GK)^{-1} \in \mathcal{C}^{n_y \times n_y}, S_i = (I + KG)^{-1} \in \mathcal{C}^{n_u \times n_u})$$

Internally stable $\iff T_o(s), S_o(s)G(s)$ and $K(s)S_o(s)$ stable.

Or, equivalently,:

Internally stable $\iff S_o(s)$ stable and no RHP cancellations in $G(s)K(s)$. (minimal realisations of GK & KG contain all RHP poles).

Internal stability

Consequences:

If $G(s)$ has a RHP-zero at z then (if internally stable),

$$\left. \begin{aligned} L_o(s) &= G(s)K(s) \\ T_o(s) &= G(s)K(s)(I - G(s)K(s))^{-1} \\ S_o(s)G(s) &= (I + G(s)K(s))^{-1}G(s) \\ L_i(s) &= K(s)G(s) \\ T_i(s) &= K(s)G(s)(I + K(s)G(s))^{-1} \end{aligned} \right\} \text{ have a RHP-zero at } z.$$

Feedback will not move (or remove) the RHP-zero from the closed-loop transfer functions.

Internal stability

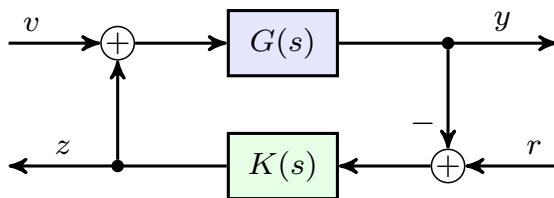
Consequences:

If $G(s)$ has a RHP-pole at p then (if internally stable),

$$\left. \begin{array}{l} L_o(s) = G(s)K(s) \\ L_i(s) = K(s)G(s) \end{array} \right\} \text{have a RHP-pole at } p,$$

$$\left. \begin{array}{l} S_o(s) = (I + G(s)K(s))^{-1} \\ K(s)S_o(s) = K(s)(I + G(s)K(s))^{-1} \\ S_i(s) = (I + K(s)G(s))^{-1} \end{array} \right\} \text{have a RHP-zero at } p.$$

Stabilizing controllers



$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} S_o G & T_o \\ -T_i & S_i K \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

Stable plant case:

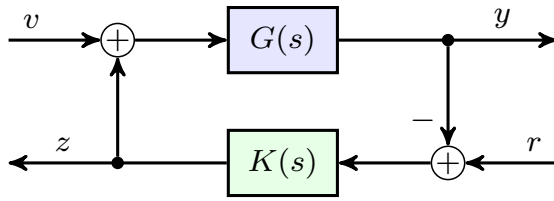
Define:

$$Q(s) = K(s)(I + G(s)K(s))^{-1}$$

Then,

$$\left. \begin{array}{l} S_o G = (I + GK)^{-1} G = (I - GQ)G \\ T_o = GK(I + GK)^{-1} = GQ \\ T_i = KG(I + KG)^{-1} = QG \\ S_i K = (I + KG)^{-1} K = Q \end{array} \right\} \text{are stable if } Q \text{ is stable.}$$

Stabilizing controllers



$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} S_o G & T_o \\ -T_1 & S_1 K \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

Stable plant case:

The converse is true:

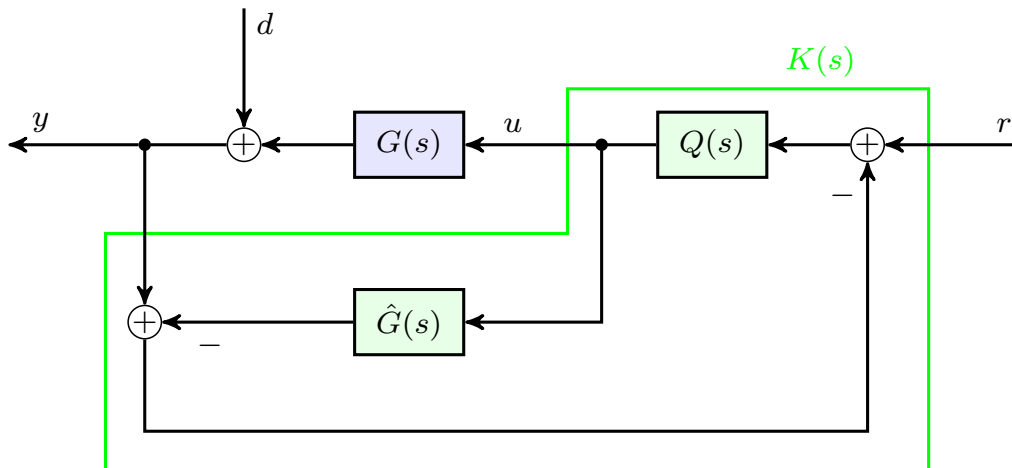
For every stabilizing controller $K(s)$,

$$Q(s) = K(s)(I + G(s)K(s))^{-1}, \quad \text{is also stable.}$$

This is a parameterisation of all stabilizing controllers.

Q -parameterisation or Youla parameterisation.

Internal model control (IMC)



Assume that $G(s)$ is stable and a perfect model: $G(s) = \hat{G}(s)$

$$y = d + Gu = \underbrace{GQ}_{T_o} r + \underbrace{(I - GQ)}_{S_o} d$$

$$u = [(I - QG)^{-1}Q \quad -(I - QG)^{-1}Q] \begin{bmatrix} r \\ y \end{bmatrix} = [K \quad -K] \begin{bmatrix} r \\ y \end{bmatrix}$$

IMC design (for stable $G(s)$)

$$Q = K(I + GK)^{-1}, \quad K = (I - QG)^{-1}Q$$

Closed-loop in linear in Q :

$$T(s) = G(s)Q(s)$$

Design approach:

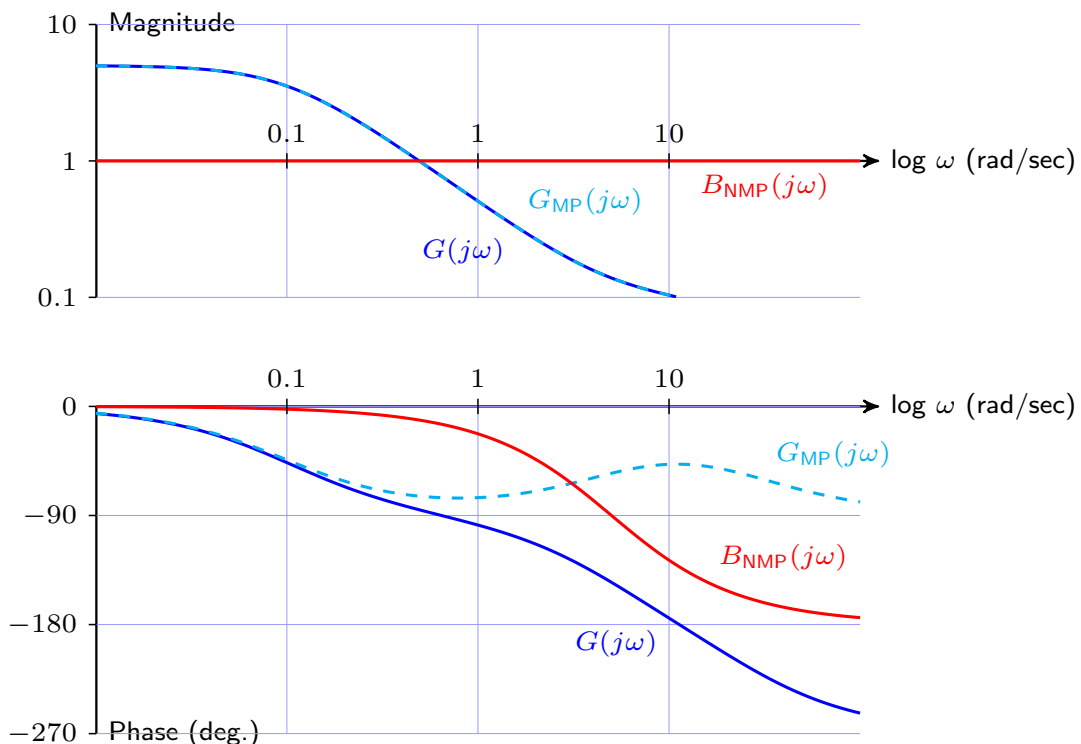
$$Q(s) = G(s)^{-1}T_{\text{ideal}}(s)$$

$$\text{or if } G(s) = B_{\text{NMP}}(s)G_{\text{MP}}(s), \quad Q(s) = G_{\text{MP}}(s)^{-1}T_{\text{ideal}}(s).$$

- ▶ Cannot invert non-minimum phase parts of $G(s)$.
- ▶ Relative degree of $T_{\text{ideal}}(s) \geq$ relative degree of $G_{\text{MP}}(s)$ makes $Q(s)$ proper.

IMC design example

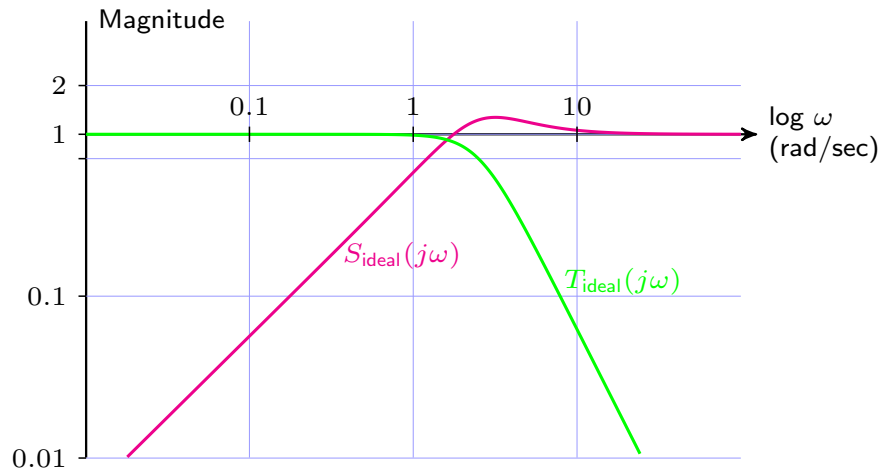
$$G(s) = \frac{5}{(1+5s)} \frac{(1-s/5)}{(1+s/25)} = \underbrace{\frac{(1-s/5)}{(1+s/5)}}_{B_{\text{NMP}}(s)} \underbrace{\frac{5}{(1+5s)} \frac{(1+s/5)}{(1+s/25)}}_{G_{\text{MP}}(s)}$$



IMC design example

Select a desired closed-loop transfer function:

$$T_{\text{ideal}}(s) = \frac{\omega_c^2}{(s^2 + \sqrt{2}\omega_c s + \omega_c^2)}, \quad \omega_c = 2.5, \quad S_{\text{ideal}}(s) = 1 - T_{\text{ideal}}(s).$$



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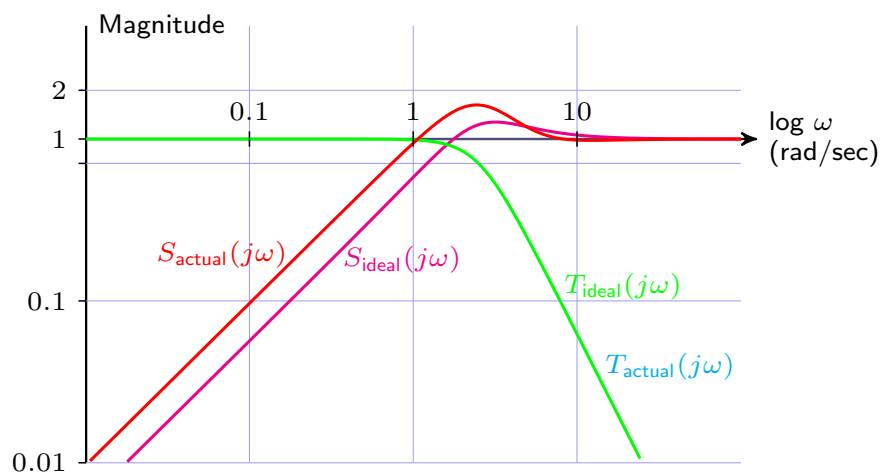
IMC design example

Invert $G_{\text{MP}}(s)$ to get $Q(s)$.

$$Q(s) = G_{\text{MP}}(s)^{-1}T_{\text{ideal}}(s) = \frac{(1 + 5s)}{5} \frac{(1 + s/25)}{(1 + s/5)} \frac{\omega_c^2}{(s^2 + \sqrt{2}\omega_c s + \omega_c^2)}$$

The actual closed-loop, $T(s)$, is:

$$T(s) = G(s)Q(s) = B_{\text{NMP}}(s)T_{\text{ideal}}(s) = \frac{(1 - s/5)}{(1 + s/5)} \frac{\omega_c^2}{(s^2 + \sqrt{2}\omega_c s + \omega_c^2)}.$$

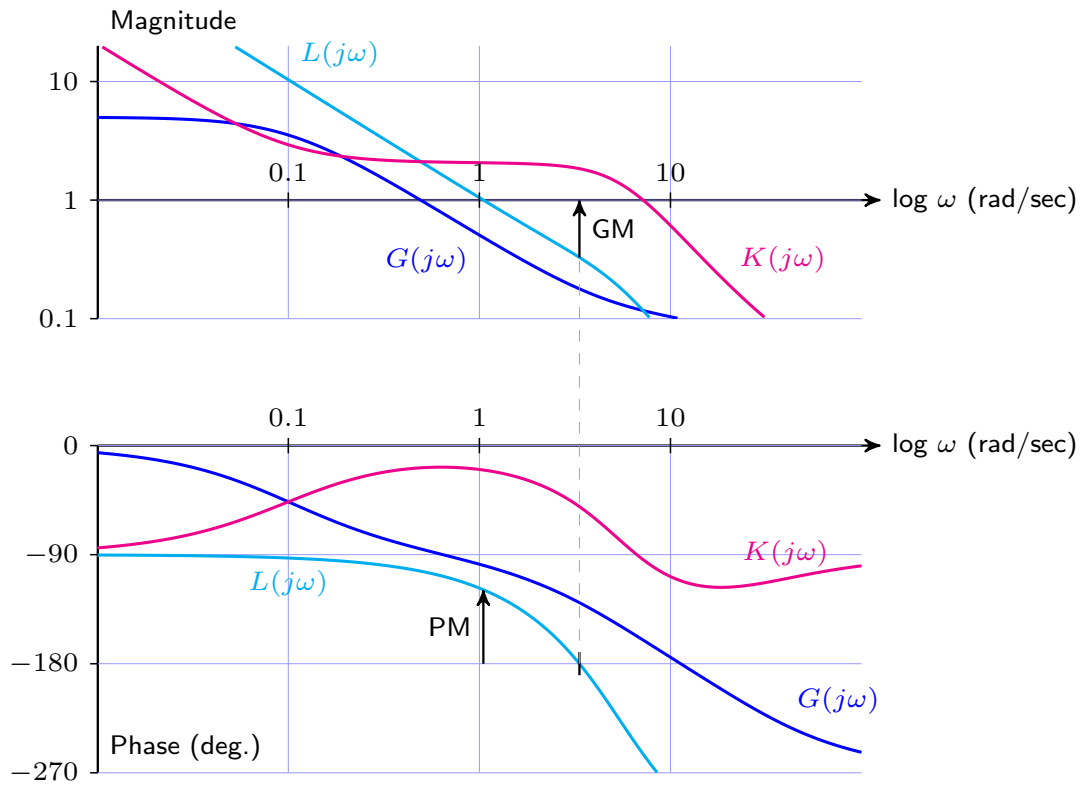


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IMC design example

Controller: $K(s) = (I - Q(s)G(s))^{-1}Q(s)$ (5th order controller)

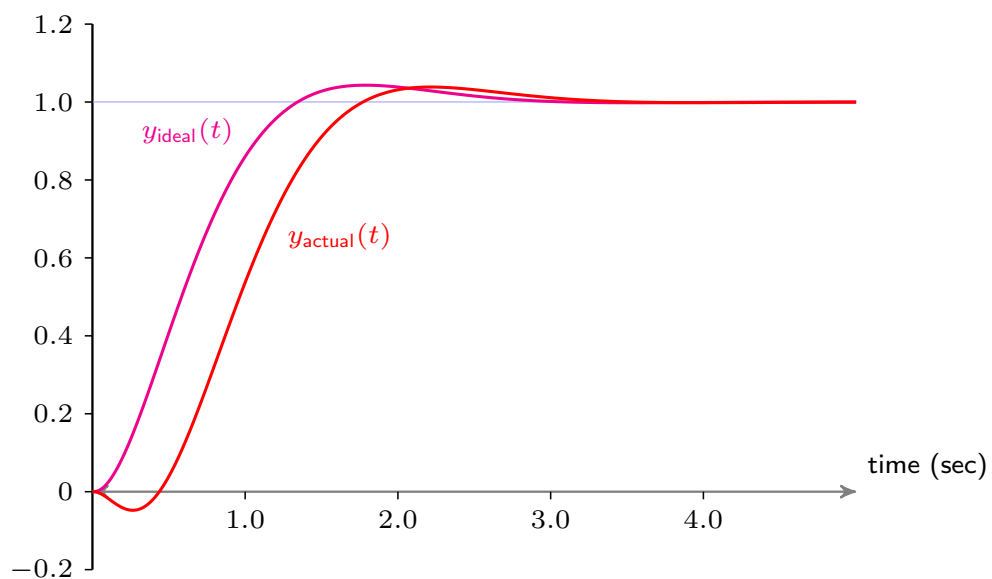


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IMC design example

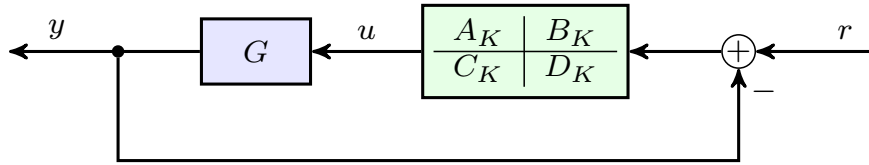
Closed-loop unit step responses:



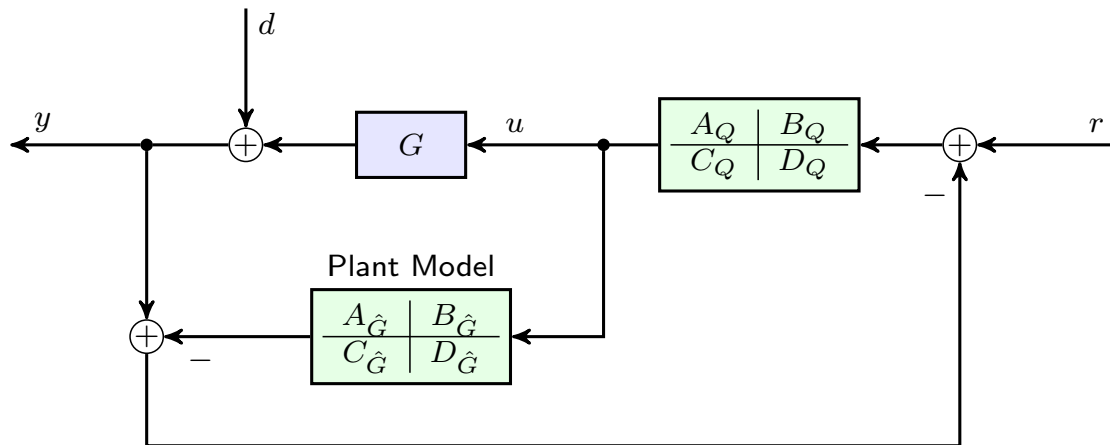
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IMC implementation



Or ...

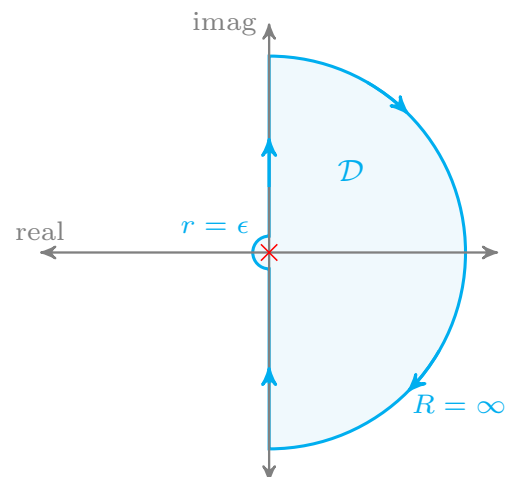
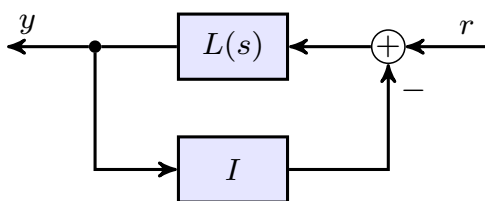


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MIMO Nyquist stability analysis

For a minimal $L(s)$,



Closed-loop exponential stability

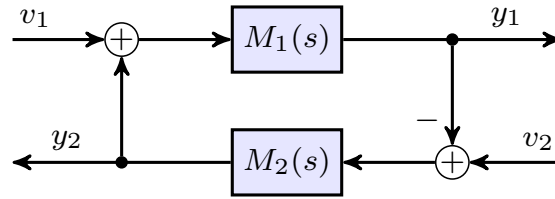
If and only if,

- i) $\det(I + L(s)) \neq 0$, for all $s \in \mathcal{D}$
- ii) The number of CCW encirclements of the origin by $\det(I + L(s))$, as s traverses the boundary of \mathcal{D} , is equal to the number of unstable poles in $L(s)$.

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Small gain theorem



A sufficient condition for stability

Given $M_1(s)$ and $M_2(s)$ stable and minimal with,

$$\|M_1(s)\| = \gamma_1 \quad \text{and} \quad \|M_2(s)\| = \gamma_2$$

If $\gamma_1\gamma_2 < 1$ then

the closed-loop interconnection is stable.

This holds for any induced norm (with the same norm for input and output signals).

\mathcal{H}_∞ norm

The \mathcal{H}_∞ norm is a measure of the “size” or “gain” of a system.

If $y(s) = G(s)u(s)$ (and stable) then,

$$\|G(s)\|_{\mathcal{H}_\infty} := \sup_{u(s) \neq 0} \frac{\|y(s)\|_2}{\|u(s)\|_2} \quad (\text{induced norm with the space})$$

$$= \sup_{u(s) \neq 0} \frac{\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} y(j\omega)^T y(j\omega) d\omega \right)^{1/2}}{\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} u(j\omega)^T u(j\omega) d\omega \right)^{1/2}}$$

$$= \max_{\omega} \bar{\sigma}(G(j\omega)) = \|G(s)\|_\infty \quad (\text{alternative notation})$$

\mathcal{H}_∞ is the set of stable, \mathcal{H}_∞ -norm bounded transfer functions.

\mathcal{H}_2 norm

Another measure of the “size” or “gain” of a system.

$$\|G(s)\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}(G(j\omega)^* G(j\omega)) d\omega \right)^{1/2}$$

The integrand is the Frobenius norm squared of the frequency response:

$$\text{trace}(G(j\omega)^* G(j\omega)) = \sum_{i,j} |G_{ij}(j\omega)|^2 = \|G(j\omega)\|_F^2.$$

Via Parseval's theorem:

$$\|G(s)\|_{\mathcal{H}_2} = \|g(t)\|_{\mathcal{H}_2} = \left(\int_0^{\infty} \text{trace}(g(\tau)^T g(\tau)) d\tau \right)^{1/2}$$

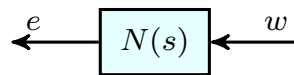
\mathcal{H}_2 norm

For state-space representations:

$$\begin{aligned} \|G(s)\|_{\mathcal{H}_2}^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}(G(j\omega)^* G(j\omega)) d\omega \\ &= \int_0^{\infty} \text{trace}(B^T e^{A^T \tau} C^T C e^{A\tau} B) d\tau \\ &= \text{trace}(B^T W_o B) \quad (W_o : \text{observability Grammian}) \\ &= \text{trace}(C W_c C^T) \quad (W_c : \text{controllability Grammian}) \end{aligned}$$

(writing $\|G(s)\|_{\mathcal{H}_2}^2$ avoids square roots)

Nominal performance norm tests



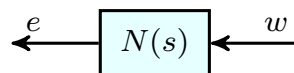
$\|N(s)\|_{\mathcal{H}_2} < 1$ implies:

- ▶ If $w(t) = \delta(t)$, then $\|e(t)\|_2 < 1$.
- ▶ If $\|w(t)\|_2 < 1$, then $\max_t |e(t)| < 1$.
- ▶ If $w(t)$ is unit variance white noise, the $\text{var}(e(t)) < 1$.

$\|N(s)\|_{\mathcal{H}_\infty} < 1$ implies:

- ▶ If $w(t) = \sin(\omega t)$ then, $\max_t |e(t)| < 1$.
- ▶ If $\|w(t)\|_2 < 1$ then, $\|e(t)\|_2 < 1$.

System norm comparison



\mathcal{H}_2 norm

- ▶ Useful nominal performance measure.
- ▶ Linear quadratic (LQ) design methods use this norm.
- ▶ Minimizes “square errors”.

\mathcal{H}_∞ norm

- ▶ Useful nominal performance measure.
- ▶ Minimizes “worst-case” errors.
- ▶ Induced norm: small-gain applies.
- ▶ Very useful for robustness analysis.

Notes and references

Skogestad & Postlethwaite (2nd Ed.)

Internal stability: section 4.7

Stabilizing controllers: section 4.8

Stability analysis: section 4.9

System norms: section 4.10

IMC design

Robust Process Control, Manfred Morari & Evangelos Zafiriou, Prentice-Hall, 1989. (Chapters 3–6).

MIMO Nyquist criterion

“On the Generalized Nyquist Stability Criterion,” C.A. Desoer and Y.-T. Wang, *IEEE Trans. Auto. Control*, v. 25, no. 2, pp. 187–196, 1980.