Vector Geometry Problems and Solutions

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Preface

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1 Problems

Exercise 1. *Prove with the aid of the scalar product:*
a) Pythagoras' theorem.
b) diagonals of a rhombus are perpendicular.

Exercise 2. *Determine the point P such that $\angle APB = 90°$.*
a) $A = (4, 2, 8)$, $B = (6, 5, -3)$, P on the y-axis.
b) $A = (3, -2, 1)$, $B = (-1, 4, 3)$, P on the x-axis.
c) $A = (3, -4, 6)$, $B = (8, 8, 8)$, P on the z-axis.
d) $A = (1, -2, 4)$, $B = (2, 5, 1)$, P on the x-axis.

Exercise 3. *The vectors*

$$\vec{a} = \begin{pmatrix} x \\ -2 \\ 6 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 6 \\ z \end{pmatrix}$$

form the sides of a square. Calculate the area of this square.

Exercise 4. *Determine the Cartesian equation of the following lines:*
a) Passes through $A(4, 7)$ with gradient $m = 3$.
b) Passes through $A(-2, 1)$ and $B(5, 3)$.
c) Passes through $A(5, -2)$ and intersects the y-axis at $y = 4$.
d) Passes through $A(-3, -7)$ and is parallel to the y-axis.
e) Passes through $A(8, -5)$ and is parallel to the x-axis.

Exercise 5. *Determine whether the following lines l_1 and l_2 are parallel:*
a) $l_1 : 3x - 4y + 5 = 0, \quad l_2 : \frac{3}{4}x - 4 = 0$
b) $l_1 : 3x - 5y + 4 = 0, \quad l_2 : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} + t \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

Exercise 6. *Given two parallel lines, find the Cartesian equation of the line lying halfway between them:*
a) $l_1 : 3x - 2y + 4 = 0$, $l_2 : 6x - 4y - 3 = 0$

Exercise 7. *Calculate the point and angle of intersection of the lines:*
a) $l_1 : 2x - 3y - 3 = 0, \quad l_2 : 3x + 2y - 24 = 0$

Exercise 8. *Find the coordinates of the orthocentre H of the triangle ABC:*
a) $A(0, 0)$, $B(12, 6)$, $C(4, 12)$
b) $A(-5, -1)$, $B(11, 9)$, $C(5, 9)$

Exercise 9. *Reflect the point P in the line l and calculate the coordinates of the reflected point P':*
a) $l : 3x - 5y - 5 = 0$, $P(2, 7)$

Exercise 10. *Reflect the line l_1 in the line l_2 and calculate the Cartesian equation of the reflected line l_1':*
a) $l_1 : 2x + y - 12 = 0, \quad l_2 : 2x + 3y - 12 = 0$

Exercise 11. *Find a vector equation of the line that passes through:*
a) $A(-4, 0, 3)$ and $B(3, 2, 5)$
b) $A(2, -1, 5)$ and intersects the x-axis at $x = 5$
c) $A(4, 3, -3)$ and is parallel to the z-axis
d) $A(7, 5, 3)$ and is parallel to the y-axis

Exercise 12. *Given the points $A(-2,1,5)$ and $B(4,2,-5)$, find a vector equation of the line that passes through the midpoint of the segment $\overline{AB}$ and is parallel to the x-axis.*

Exercise 13. *The points $A(4,1,3)$, $B(2,-4,3)$, and $C(7,4,1)$ are the vertices of a triangle. Find a vector equation of the line that passes through the point $P(1,0,1)$ and the centroid of the triangle ABC.*

Exercise 14. *Are the points $A(5,4,2)$, $B(0,-11,-7)$, and $C(7.5,11.5,7)$ on the line*

$$l: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -4 \end{pmatrix} + t \begin{pmatrix} 1.5 \\ 3 \\ 2 \end{pmatrix}?$$

Exercise 15. *Describe the particular position of the line l:*

$$l: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Exercise 16. *Determine the Cartesian equation of the plane defined by the points A, B, C:*
a) $A(4,1,7),\ B(3,-1,2),\ C(2,0,0)$
b) $A(4,-2,-2),\ B(7,2,4),\ C(0,-5,-3)$
c) $A(2,5,4),\ B(7,0,3),\ C(-8,-5,2)$

Exercise 17. *Determine the Cartesian equation of the plane defined by the point $P(0,-2,0)$ and the line*

$$l: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Exercise 18. *A cube is given. Let A and B be the midpoints of two edges that are orthogonal to each other, and let C be the intersection of the diagonals of a side of the cube. Determine the Cartesian equation of the plane through A, B, and C.*

Exercise 19. *Determine the Cartesian equation of the following planes:*
a) Plane through points $A(0,5,3), B(2,-1,-1), C(-1,6,5)$
b) Plane through point $P(4,-1,1)$ and the line

$$g: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ -2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}.$$

Exercise 20. *Determine the Cartesian equation of the following planes:*
a) The plane containing the lines

$$l_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \quad l_2: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

b) The plane passing through the point $P(4,-2.5,1)$ and parallel to the xz-plane.

Exercise 21. *Describe the particular position of the following planes:*
a) $\mathbb{P}: 3x - 7y + 3 = 0$
b) $\mathbb{P}: 2y - 5z + 9 = 0$
c) $\mathbb{P}: 4x - 5 = 0$

Exercise 22. *Prove that the lines l_1 and l_2 intersect and determine the Cartesian equation of the plane containing them.*
a)

$$l_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} + s \begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix}, \qquad l_2: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$$

b)

$$l_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 8 \end{pmatrix} + s \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \qquad l_2: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -8 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

Exercise 23. *Calculate the intercepts with the coordinate axes of the plane $\mathbb{P}$.*
a) $\mathbb{P}: 3x - y + 2z - 6 = 0$
b) $\mathbb{P}: 3x + 9y - 3z + 36 = 0$
c) $\mathbb{P}: 3x - 5z + 60 = 0$

Exercise 24. *Determine the Cartesian equation of the plane with intercepts*

$$x = a, \quad y = b, \quad z = c.$$

Then divide the equation by abc.

Exercise 25. *A plane is given by its axis intercepts. Determine its Cartesian equation.*
a) $a = 2,\ b = 3,\ c = 3$
b) $a = 8,\ b = -6,\ c = 1$
c) $a = -9,\ b = 5,\ c = 3$
d) $a = b = -14,\ c = 6$
e) $a = 7$

Exercise 26. *Do the points A and B lie in the plane $\mathcal{P}$?*
a)

$$\mathcal{P}: 3x - 2y - 5z + 11 = 0, \quad A(5, -2, 6), \quad B(4, 6, 2)$$

b)

$$\mathcal{P}: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + u \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + v \begin{pmatrix} 4 \\ -4 \\ -3 \end{pmatrix}, \quad A(2, 4, 1), \quad B(4, 6, 2).$$

Exercise 27. *Determine whether the points*

$$A(2, 1, 10), \quad B(0, 4, -3), \quad C(-1, 0, 5), \quad D(3, 5, 2)$$

are vertices of a quadrilateral.

Exercise 28. *Find the coordinates of the point of intersection of the three planes*

$$\mathcal{P}_1: 3x - 2y + z - 8 = 0, \quad \mathcal{P}_2: x + y - 3z + 4 = 0, \quad \mathcal{P}_3: 4x + 3y - 5z - 8 = 0.$$

Exercise 29. *The plane*

$$\mathcal{P}: 9x + 16y + cz - 144 = 0$$

intersects the coordinate axes at the points A, B, and C. Let O be the origin. Find the value of c given that the pyramid $OABC$ has volume

$$V = 384.$$

Exercise 30. *Find the point where the line $\updownarrow$ intersects the plane $\mathcal{P}$:*
a)

$$\updownarrow: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -10 \\ 8 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}, \quad \mathcal{P}: 3x - y + 4z - 15 = 0$$

b)

$$\updownarrow: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 9 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}, \quad \mathcal{P}: 4y - 3z + 27 = 0$$

Exercise 31. *Consider the triangle ABC with vertices*

$$A(0,1,5), \quad B(-7,3,-5), \quad C(5,-3,7),$$

and let $h_A = \overline{AD}$ be the altitude from A to side BC. Find the coordinates of D and the area of the triangle ABC.

Exercise 32. *Determine the cartesian equation of the plane which contains the points*

$$P(2,0,1), \quad Q(-3,-4,2)$$

and is normal to the plane

$$\mathcal{P}: 3x + y + 7z - 2 = 0.$$

Exercise 33. *Reflect the point*

$$P(0,4,-5)$$

in the plane

$$\mathcal{P}: 4x - 3y + z + 4 = 0$$

and find the coordinates of the reflected point P'.

Exercise 34. *Derive the formula for the reflection of a point P across a line*

$$\ell: \vec{r} = \vec{A} + t\vec{v}, \quad t \in \mathbb{R},$$

where $\vec{v}$ is the direction vector of the line.

Exercise 35. *Derive the formula for the reflection of a point P across the plane*

$$\mathcal{P}: ax + by + cz + d = 0.$$

Exercise 36. *Point $P' = (0,0,7)$ is the reflection of $P = (4,3,-2)$ across a plane $\mathcal{P}$. Determine the Cartesian equation of $\mathcal{P}$.*

Exercise 37. *The line $\updownarrow$ is reflected in the plane $\mathcal{P}$. Determine a vector equation of the reflected line $\updownarrow'$:*

$$\updownarrow: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix}, \quad \mathcal{P}: x - 3y - 2z + 42 = 0.$$

Exercise 38. *A ray of light passes through $P = (7,-7,4)$ and is reflected in the plane*

$$\mathcal{P}: 5x - 2y + 3z - 23 = 0.$$

Point $Q = (7,-1,8)$ lies on the reflected ray. Determine the point R on the plane $\mathcal{P}$ where the reflection occurs.

Exercise 39. *Determine the distance between a point P and a line $\updownarrow$:*
a) $P = (3, -1), \quad \updownarrow : y = 2.4x - 0.4$
b) $P = (6, 5), \quad \updownarrow : (x, y) = (-5, -2) + t(15, 8)$

Exercise 40. *Find the distance between the skew lines:*

$$\updownarrow_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 12 \\ 0 \end{pmatrix}, \quad \updownarrow_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + s \begin{pmatrix} 0 \\ 12 \\ -1 \end{pmatrix}.$$

2 Solutions

Solution. **Exercise 1.**
a) Pythagoras' theorem
Let $\vec{a}, \vec{b} \in \mathbb{R}^n$ be two nonzero vectors such that the angle between them is $\frac{\pi}{2}$. By definition of the scalar product,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \frac{\pi}{2} = 0.$$

Consider the vector

$$\vec{c} := \vec{a} + \vec{b},$$

which represents the hypotenuse of the right-angled triangle spanned by $\vec{a}$ and $\vec{b}$. Using bilinearity and symmetry of the scalar product, we compute

$$|\vec{c}|^2 = \vec{c} \cdot \vec{c} = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{a} \cdot \vec{a} + 2\,\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}.$$

Since $\vec{a} \cdot \vec{b} = 0$, it follows that

$$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2.$$

This proves Pythagoras' theorem.

□

b) Diagonals of a rhombus are perpendicular
Let $\vec{a}, \vec{b} \in \mathbb{R}^n$ be two nonzero vectors such that

$$|\vec{a}| = |\vec{b}|.$$

Assume that the angle between $\vec{a}$ and $\vec{b}$ is $\theta \in (0, \pi)$. Then $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a rhombus. The diagonals of this rhombus are represented by the vectors

$$\vec{d_1} := \vec{a} + \vec{b}, \qquad \vec{d_2} := \vec{a} - \vec{b}.$$

Using bilinearity and symmetry of the scalar product, we compute

$$\vec{d_1} \cdot \vec{d_2} = (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b}.$$

Since the scalar product is symmetric, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$, hence

$$\vec{d_1} \cdot \vec{d_2} = |\vec{a}|^2 - |\vec{b}|^2.$$

Using the assumption $|\vec{a}| = |\vec{b}|$, we conclude that

$$\vec{d_1} \cdot \vec{d_2} = 0.$$

Therefore, the diagonals of a rhombus are perpendicular.

□

Solution. **Exercise 2.**
a) $A = (4, 2, 8)$, $B = (6, 5, -3)$, P on the y-axis.
Let $P = (0, y, 0)$. Then

$$\overrightarrow{PA} = (4, 2 - y, 8), \qquad \overrightarrow{PB} = (6, 5 - y, -3).$$

The orthogonality condition is

$$\overrightarrow{PA} \cdot \overrightarrow{PB} = 4 \cdot 6 + (2 - y)(5 - y) + 8 \cdot (-3) = 0.$$

Expanding gives

$$24 + (10 - 7y + y^2) - 24 = y^2 - 7y + 10 = 0,$$

which implies

$$y = 2 \quad \text{or} \quad y = 5.$$

Hence

$$P = (0, 2, 0) \quad \text{or} \quad P = (0, 5, 0).$$

b) $A = (3, -2, 1),\ B = (-1, 4, 3),\ P$ on the x-axis.
Let $P = (x, 0, 0)$. Then

$$\overrightarrow{PA} = (3 - x, -2, 1), \qquad \overrightarrow{PB} = (-1 - x, 4, 3).$$

Orthogonality condition:

$$(3 - x)(-1 - x) + (-2) \cdot 4 + 1 \cdot 3 = 0.$$

Expanding gives

$$x^2 - 2x - 8 = 0 \quad \Rightarrow \quad x = 4 \text{ or } x = -2.$$

Hence

$$P = (4, 0, 0) \quad \text{or} \quad P = (-2, 0, 0).$$

c) $A = (3, -4, 6),\ B = (8, 8, 8),\ P$ on the z-axis.
Let $P = (0, 0, z)$. Then

$$\overrightarrow{PA} = (3, -4, 6 - z), \qquad \overrightarrow{PB} = (8, 8, 8 - z).$$

Orthogonality condition:

$$3 \cdot 8 + (-4) \cdot 8 + (6 - z)(8 - z) = 0.$$

Expanding gives

$$z^2 - 14z + 40 = 0 \quad \Rightarrow \quad z = 4 \text{ or } z = 10.$$

Hence

$$P = (0, 0, 4) \quad \text{or} \quad P = (0, 0, 10).$$

d) $A = (1, -2, 4),\ B = (2, 5, 1),\ P$ on the x-axis.
Let $P = (x, 0, 0)$. Then

$$\overrightarrow{PA} = (1 - x, -2, 4), \qquad \overrightarrow{PB} = (2 - x, 5, 1).$$

Orthogonality condition:

$$(1 - x)(2 - x) + (-2) \cdot 5 + 4 \cdot 1 = 0.$$

Expanding gives

$$x^2 - 3x - 4 = 0 \quad \Rightarrow \quad x = 4 \text{ or } x = -1.$$

Hence

$$P = (4, 0, 0) \quad \text{or} \quad P = (-1, 0, 0).$$

Solution. **Exercise 3.**

Let

$$\vec{a} = \begin{pmatrix} x \\ -2 \\ 6 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 6 \\ z \end{pmatrix}$$

be the sides of a square.

Step 1: Equal lengths.

For two vectors to be adjacent sides of a square, they must have equal lengths:

$$|\vec{a}|^2 = x^2 + (-2)^2 + 6^2 = x^2 + 40,$$

$$|\vec{b}|^2 = 2^2 + 6^2 + z^2 = z^2 + 40.$$

Equating lengths gives

$$x^2 + 40 = z^2 + 40 \quad \Rightarrow \quad x^2 = z^2 \quad \Rightarrow \quad z = \pm x.$$

Step 2: Orthogonality.

The vectors must also be perpendicular:

$$\vec{a} \cdot \vec{b} = x \cdot 2 + (-2) \cdot 6 + 6 \cdot z = 2x - 12 + 6z = 0.$$

Substitute $z = x$:

$$2x - 12 + 6x = 8x - 12 = 0 \quad \Rightarrow \quad x = \frac{3}{2}, \quad z = \frac{3}{2}.$$

Substitute $z = -x$:

$$2x - 12 + 6(-x) = -4x - 12 = 0 \quad \Rightarrow \quad x = -3, \quad z = 3.$$

Step 3: Area of the square.

The area of a square with side vector $\vec{a}$ is

$$\text{Area} = |\vec{a}|^2 = x^2 + 40.$$

Case 1: $x = \frac{3}{2}$

$$\text{Area} = \left(\frac{3}{2}\right)^2 + 40 = \frac{9}{4} + 40 = \frac{169}{4}.$$

Case 2: $x = -3$

$$\text{Area} = (-3)^2 + 40 = 9 + 40 = 49.$$

Answer:

$$\boxed{\text{Area} = \frac{169}{4} \text{ or } 49}.$$

Solution. **Exercise 4.**

a) Line through $A(4, 7)$ with gradient $m = 3$.

Point-slope form:

$$y - 7 = 3(x - 4)$$

$$\Rightarrow y = 3x - 5$$

b) Line through $A(-2, 1)$ and $B(5, 3)$.

Slope:

$$m = \frac{3-1}{5-(-2)} = \frac{2}{7}$$

Equation through $A(-2,1)$:

$$y - 1 = \frac{2}{7}(x+2) \quad \Rightarrow \quad y = \frac{2}{7}x + \frac{11}{7}$$

c) Line through $A(5,-2)$ and intersects y-axis at $y=4$.

Slope using $B=(0,4)$:

$$m = \frac{4-(-2)}{0-5} = -\frac{6}{5}$$

Equation through $A(5,-2)$:

$$y + 2 = -\frac{6}{5}(x-5) \quad \Rightarrow \quad y = -\frac{6}{5}x + 4$$

d) Line through $A(-3,-7)$ parallel to y-axis.

Equation:

$$x = -3$$

e) Line through $A(8,-5)$ parallel to x-axis.

Equation:

$$y = -5$$

Solution. **Exercise 5.**

a) $l_1 : 3x - 4y + 5 = 0,\ l_2 : \frac{3}{4}x - 4 = 0$

Rewrite in slope-intercept form:

$$l_1 : 3x - 4y + 5 = 0 \quad \Rightarrow \quad y = \frac{3}{4}x + \frac{5}{4}$$

$$l_2 : \frac{3}{4}x - 4 = 0 \quad \Rightarrow \quad x = \frac{16}{3}$$

l_2 is vertical, l_1 has slope $3/4$.

Conclusion: Not parallel.

b) $l_1 : 3x - 5y + 4 = 0,\ l_2 : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} + t \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

Slope of l_1:

$$y = \frac{3}{5}x + \frac{4}{5} \quad \Rightarrow \quad m_1 = \frac{3}{5}$$

Direction vector of l_2 is $\begin{pmatrix} 5 \\ 3 \end{pmatrix} \Rightarrow m_2 = \frac{3}{5}$.

Conclusion: Parallel.

Solution. **Exercise 6.**

a) $l_1 : 3x - 2y + 4 = 0,\ l_2 : 6x - 4y - 3 = 0$

Step 1: Check parallelism

Divide l_2 by 2:

$$6x - 4y - 3 = 0 \quad \Rightarrow \quad 3x - 2y - \frac{3}{2} = 0$$

Same coefficients for x and y as $l_1 \rightarrow$ lines are parallel.

Step 2: Line halfway formula

For two parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$, the halfway line is

$$Ax + By + \frac{C_1 + C_2}{2} = 0$$

Step 3: Apply the formula

$$C = \frac{4 + (-3/2)}{2} = \frac{5}{4}$$

$$\Rightarrow \text{Halfway line: } 3x - 2y + \frac{5}{4} = 0$$

Optional: slope-intercept form

$$y = \frac{3}{2}x + \frac{5}{8}$$

Solution. **Exercise 7.**

a) $l_1 : 2x - 3y - 3 = 0, \quad l_2 : 3x + 2y - 24 = 0$

Step 1: Find intersection point

From l_1:

$$2x - 3y = 3 \quad \Rightarrow \quad y = \frac{2x - 3}{3}$$

Substitute into l_2:

$$3x + 2\left(\frac{2x - 3}{3}\right) = 24 \quad \Rightarrow \quad 13x - 6 = 72 \quad \Rightarrow \quad x = 6$$

$$y = \frac{2 \cdot 6 - 3}{3} = 3$$

Intersection point:

$$\boxed{(6, 3)}$$

Step 2: Find angle of intersection

Slope of $l_1 : y = \frac{2}{3}x - 1 \Rightarrow m_1 = \frac{2}{3}$

Slope of $l_2 : y = -\frac{3}{2}x + 12 \Rightarrow m_2 = -\frac{3}{2}$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{2}{3} - (-\frac{3}{2})}{1 + \frac{2}{3} \cdot (-\frac{3}{2})}\right| = \infty$$

Therefore, the lines are perpendicular:

$$\theta = 90^\circ$$

Solution. **Exercise 8.**

a) $A(0, 0),\ B(12, 6),\ C(4, 12)$

The orthocentre H is the intersection of the **altitudes** of the triangle.

1. **Altitude from A:** perpendicular to BC.

- Slope of BC:

$$m_{BC} = \frac{12 - 6}{4 - 12} = \frac{6}{-8} = -\frac{3}{4}$$

- Slope of altitude from A (perpendicular):

$$m_{AH} = \frac{4}{3} \quad \text{(negative reciprocal)}$$

- Equation of altitude from A through $(0,0)$:

$$y = \frac{4}{3}x$$

2. **Altitude from B:** perpendicular to AC.
- Slope of AC:

$$m_{AC} = \frac{12-0}{4-0} = 3$$

- Slope of altitude from B (perpendicular):

$$m = -\frac{1}{3}$$

- Equation of altitude from $B(12,6)$:

$$y - 6 = -\frac{1}{3}(x-12) \quad \Rightarrow \quad y = -\frac{1}{3}x + 10$$

3. **Intersection (orthocentre):**

$$\frac{4}{3}x = -\frac{1}{3}x + 10 \quad \Rightarrow \quad \frac{5}{3}x = 10 \quad \Rightarrow \quad x = 6$$

$$y = \frac{4}{3} \cdot 6 = 8$$

$$\boxed{H = (6,8)}$$

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b) $A(-5,-1),\ B(11,9),\ C(5,9)$
1. **Altitude from A:** perpendicular to BC.
- Slope of BC:

$$m_{BC} = \frac{9-9}{5-11} = 0$$

- Altitude from A is **vertical line** through $x = -5$:

$$x = -5$$

2. **Altitude from B:** perpendicular to AC.
- Slope of AC:

$$m_{AC} = \frac{9-(-1)}{5-(-5)} = \frac{10}{10} = 1$$

- Perpendicular slope: $m = -1$
- Equation of altitude from $B(11,9)$:

$$y - 9 = -1(x-11) \quad \Rightarrow \quad y = -x + 20$$

3. **Intersection (orthocentre):**

$$x = -5 \quad \Rightarrow \quad y = -(-5) + 20 = 25$$

$$\boxed{H = (-5,25)}$$

Solution. **Exercise 9.**

a) $l : 3x - 5y - 5 = 0, \quad P(2, 7)$

Step 1: General formula for reflection
Let line $l : Ax + By + C = 0$, point $P(x_0, y_0)$.
Coordinates of the reflected point $P' = (x', y')$ are:

$$x' = x_0 - \frac{2A(Ax_0 + By_0 + C)}{A^2 + B^2}, \quad y' = y_0 - \frac{2B(Ax_0 + By_0 + C)}{A^2 + B^2}$$

—

Step 2: Apply formula
Here: $A = 3, B = -5, C = -5$, $x_0 = 2, y_0 = 7$

$$D = Ax_0 + By_0 + C = 3 \cdot 2 + (-5) \cdot 7 + (-5) = 6 - 35 - 5 = -34$$

$$A^2 + B^2 = 3^2 + (-5)^2 = 9 + 25 = 34$$

$$x' = 2 - \frac{2 \cdot 3 \cdot (-34)}{34} = 2 - \frac{-204}{34} = 2 + 6 = 8$$

$$y' = 7 - \frac{2 \cdot (-5) \cdot (-34)}{34} = 7 - \frac{340}{34} = 7 - 10 = -3$$

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Step 3: Answer

$$\boxed{P' = (8, -3)}$$

Solution. **Exercise 10.**

a) $l_1 : 2x + y - 12 = 0, \quad l_2 : 2x + 3y - 12 = 0$

Step 1: Pick two points on l_1
- $x = 0 \Rightarrow y = 12 \quad \Rightarrow P_1 = (0, 12)$ - $y = 0 \Rightarrow 2x - 12 = 0 \Rightarrow x = 6 \quad \Rightarrow P_2 = (6, 0)$

Step 2: Reflect points across $l_2 : 2x + 3y - 12 = 0$
- $A = 2, B = 3, C = -12$

$$x' = x_0 - \frac{2A(Ax_0 + By_0 + C)}{A^2 + B^2}, \quad y' = y_0 - \frac{2B(Ax_0 + By_0 + C)}{A^2 + B^2}$$

- Reflect $P_1 = (0, 12)$:

$$D = 2 \cdot 0 + 3 \cdot 12 - 12 = 24, \quad A^2 + B^2 = 13$$

$$x_1' = 0 - \frac{2 \cdot 2 \cdot 24}{13} = -\frac{96}{13}, \quad y_1' = 12 - \frac{2 \cdot 3 \cdot 24}{13} = -\frac{12}{13}$$

$$P_1' = \left(-\frac{96}{13}, -\frac{12}{13}\right)$$

- Reflect $P_2 = (6, 0)$:

$$D = 2 \cdot 6 + 3 \cdot 0 - 12 = 0$$

$$P_2' = (6, 0)$$

Step 3: Equation of reflected line

- Slope:

$$m = \frac{0-(-12/13)}{6-(-96/13)} = \frac{12/13}{174/13} = \frac{2}{29}$$

- Equation through $P_2' = (6,0)$:

$$y - 0 = \frac{2}{29}(x-6) \quad \Rightarrow \quad y = \frac{2}{29}x - \frac{12}{29}$$

- Cartesian form:

$$29y - 2x + 12 = 0 \quad \Rightarrow \quad \boxed{l_1' : 2x - 29y - 12 = 0}$$

Solution. **Exercise 11.**

a) $A(-4,0,3)$, $B(3,2,5)$

A direction vector of the line is

$$\vec{d} = \overrightarrow{AB} = (3+4,\, 2-0,\, 5-3) = (7,2,2).$$

Hence, a vector equation of the line is

$$\vec{x}(t) = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 7 \\ 2 \\ 2 \end{pmatrix}, \quad t \in \mathbb{R}.$$

b) $A(2,-1,5)$, intersects the x-axis at $x = 5$

The point of intersection with the x-axis is

$$B = (5,0,0).$$

A direction vector is

$$\vec{d} = \overrightarrow{AB} = (5-2,\, 0+1,\, 0-5) = (3,1,-5).$$

Thus, the vector equation of the line is

$$\vec{x}(t) = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}, \quad t \in \mathbb{R}.$$

c) $A(4,3,-3)$, parallel to the z-axis

A direction vector parallel to the z-axis is

$$\vec{d} = (0,0,1).$$

Therefore, the vector equation of the line is

$$\vec{x}(t) = \begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

d) $A(7,5,3)$, parallel to the y-axis

A direction vector parallel to the y-axis is

$$\vec{d} = (0,1,0).$$

Hence, the vector equation of the line is

$$\vec{x}(t) = \begin{pmatrix} 7 \\ 5 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Solution. **Exercise 12.**

First, compute the midpoint M of the segment $\overline{AB}$.

$$M = \frac{1}{2}(A+B) = \frac{1}{2}((-2,1,5)+(4,2,-5)) = \frac{1}{2}(2,3,0) = (1,\tfrac{3}{2},0).$$

A direction vector parallel to the x-axis is

$$\vec{d} = (1,0,0).$$

Therefore, a vector equation of the required line is

$$\vec{x}(t) = \begin{pmatrix} 1 \\ \frac{3}{2} \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Solution. **Exercise 13.**

First, compute the centroid G of the triangle ABC. By definition,

$$G = \frac{1}{3}(A+B+C).$$

Thus,

$$G = \frac{1}{3}((4,1,3)+(2,-4,3)+(7,4,1)) = \frac{1}{3}(13,1,7) = \left(\tfrac{13}{3},\tfrac{1}{3},\tfrac{7}{3}\right).$$

A direction vector of the required line is

$$\vec{d} = \overrightarrow{PG} = \left(\tfrac{13}{3}-1,\tfrac{1}{3}-0,\tfrac{7}{3}-1\right) = \left(\tfrac{10}{3},\tfrac{1}{3},\tfrac{4}{3}\right).$$

Therefore, a vector equation of the line is

$$\vec{x}(t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} \frac{10}{3} \\ \frac{1}{3} \\ \frac{4}{3} \end{pmatrix}, \quad t \in \mathbb{R}.$$

Solution. **Exercise 14.**

The vector equation of the line is

$$(x,y,z) = (2,-5,-4) + t(1,3,2), \quad t \in \mathbb{R}.$$

Thus,

$$x = 2+t, \quad y = -5+3t, \quad z = -4+2t.$$

Point $A(5,4,2)$:
From $x = 2+t = 5$ we obtain $t = 3$. Then

$$y = -5+3\cdot 3 = 4, \qquad z = -4+2\cdot 3 = 2.$$

Hence, point A lies on the line l.

Point $B(0,-11,-7)$:
From $x = 2+t = 0$ we obtain $t = -2$. Then

$$y = -5+3(-2) = -11, \qquad z = -4+2(-2) = -8 \neq -7.$$

Thus, point B does not lie on the line l.

Point $C(7.5, 11.5, 7)$**:**

From $x = 2 + t = 7.5$ we obtain $t = 5.5$. Then

$$y = -5 + 3(5.5) = 11.5, \qquad z = -4 + 2(5.5) = 7.$$

Hence, point C lies on the line l.

Therefore, the points A and C lie on the line l, whereas point B does not.

Solution. **Exercise 15.**

The line l has the parametric representation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

From this representation, we obtain

$$x = 0, \qquad y = 0, \qquad z = 7 + t.$$

Thus, all points of the line satisfy $x = 0$ and $y = 0$, while z is arbitrary. Therefore, the line l coincides with the z-axis, shifted such that it passes through the point $(0, 0, 7)$.

In particular, the line is parallel to the z-axis and perpendicular to the xy-plane.

Solution. **Exercise 16.**

Method: The Cartesian equation of a plane through points A, B, C is

$$\vec{n} \cdot (\vec{x} - \vec{A}) = 0,$$

where $\vec{n}$ is a normal vector to the plane, given by

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}.$$

a) $A(4, 1, 7),\ B(3, -1, 2),\ C(2, 0, 0)$

$$\overrightarrow{AB} = \begin{pmatrix} 3-4 \\ -1-1 \\ 2-7 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -5 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} 2-4 \\ 0-1 \\ 0-7 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -7 \end{pmatrix}$$

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -2 \cdot (-7) - (-5)(-1) \\ -(-1 \cdot (-7) - (-5)(-2)) \\ -1 \cdot (-1) - (-2)(-2) \end{pmatrix} = \begin{pmatrix} 14 - 5 \\ -(7 - 10) \\ 1 - 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ -3 \end{pmatrix}$$

Equation of the plane:

$$9(x - 4) + 3(y - 1) - 3(z - 7) = 0 \quad \Rightarrow \quad 9x + 3y - 3z - 24 = 0$$

—

b) $A(4, -2, -2),\ B(7, 2, 4),\ C(0, -5, -3)$

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} -4 \\ -3 \\ -1 \end{pmatrix}$$

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 4 \cdot (-1) - 6(-3) \\ -(3(-1) - 6(-4)) \\ 3(-3) - 4(-4) \end{pmatrix} = \begin{pmatrix} -4 + 18 \\ -(-3 + 24) \\ -9 + 16 \end{pmatrix} = \begin{pmatrix} 14 \\ -21 \\ 7 \end{pmatrix}$$

Equation of the plane:

$$14(x - 4) - 21(y + 2) + 7(z + 2) = 0 \quad \Rightarrow \quad 14x - 21y + 7z - 105 = 0$$

—

c) $A(2, 5, 4),\ B(7, 0, 3),\ C(-8, -5, 2)$

$$\overrightarrow{AB} = \begin{pmatrix} 5 \\ -5 \\ -1 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} -10 \\ -10 \\ -2 \end{pmatrix}$$

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -5(-2) - (-1)(-10) \\ -(5(-2) - (-1)(-10)) \\ 5(-10) - (-5)(-10) \end{pmatrix} = \begin{pmatrix} 10 - 10 \\ -(-10 - 10) \\ -50 - 50 \end{pmatrix} = \begin{pmatrix} 0 \\ 20 \\ -100 \end{pmatrix}$$

We can simplify by dividing by 20:

$$\vec{n} = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}$$

Equation of the plane:

$$0(x - 2) + 1(y - 5) - 5(z - 4) = 0 \quad \Rightarrow \quad y - 5 - 5z + 20 = 0 \quad \Rightarrow \quad y - 5z + 15 = 0$$

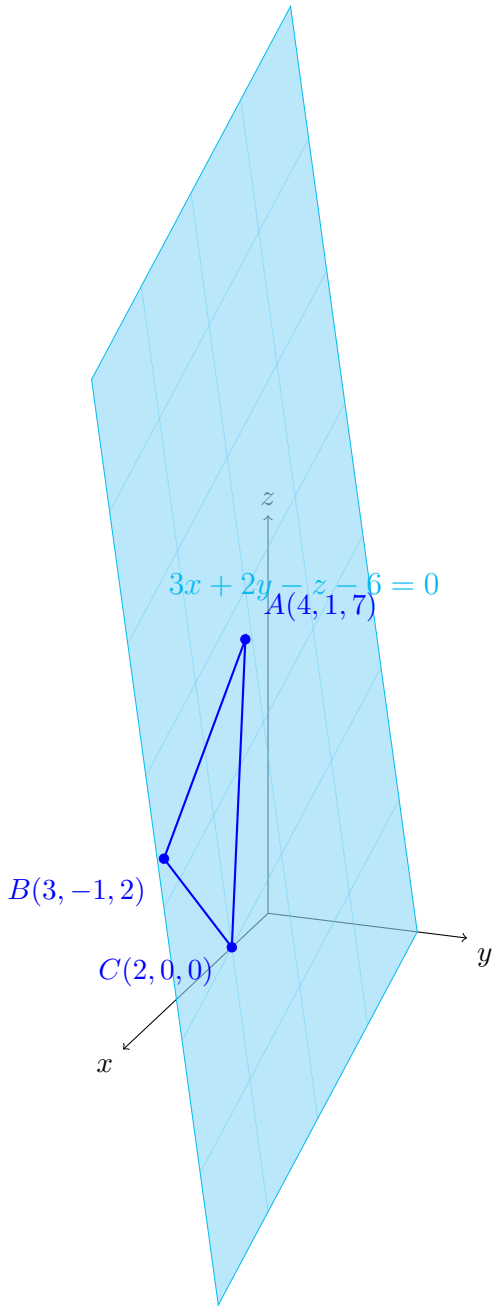


Figure 1: Plane passing through points A, B, C (Exercise 16a)

Solution. **Exercise 17.**

The plane passes through the point $P(0, -2, 0)$ and the line l. A vector lying on the plane is the direction vector of the line:

$$\vec{d} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Another vector lying on the plane can be obtained from P to a point on the line, e.g., the point $Q = (2, -1, 0)$:

$$\vec{v} = \overrightarrow{PQ} = \begin{pmatrix} 2 - 0 \\ -1 - (-2) \\ 0 - 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

A normal vector of the plane is

$$\vec{n} = \vec{d} \times \vec{v}.$$

Compute the cross product:

$$\vec{n} = \begin{pmatrix} (-1)\cdot 0 - 1\cdot 1 \\ -(1\cdot 0 - 1\cdot 2) \\ 1\cdot 1 - (-1)\cdot 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}.$$

Therefore, the Cartesian equation of the plane is

$$\vec{n}\cdot(\vec{x} - \vec{P}) = 0 \quad \Rightarrow \quad (-1)(x-0) + 2(y+2) + 3(z-0) = 0.$$

Simplify:

$$-x + 2y + 4 + 3z = 0 \quad \Rightarrow \quad -x + 2y + 3z + 4 = 0.$$

Hence, the Cartesian equation of the plane is

$$\boxed{-x + 2y + 3z + 4 = 0}.$$

Solution. **Exercise 18.**

Let us choose coordinates for simplicity:

$$A = (0.5, 0, 0), \quad B = (0, 0.5, 0), \quad C = (0.5, 0.5, 1)$$

Compute vectors in the plane:

$$\vec{AB} = B - A = (-0.5, 0.5, 0), \quad \vec{AC} = C - A = (0, 0.5, 1)$$

A normal vector to the plane is

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{pmatrix} (-0.5)(1) - (0)(0.5) \\ -((-0.5)(1) - 0\cdot 0) \\ (-0.5)(0.5) - (0.5)(0) \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \\ -0.25 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

Equation of the plane through A:

$$\vec{n}\cdot(\vec{x} - \vec{A}) = 0 \quad \Rightarrow \quad 2(x-0.5) + 2(y-0) - 1(z-0) = 0$$

Simplify:

$$2x + 2y - z - 1 = 0$$

Hence, the Cartesian equation of the plane is

$$\boxed{2x + 2y - z - 1 = 0}.$$

Solution. **Exercise 19.**

Let

$$\vec{d} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

be the common direction vector of both parallel lines.

A vector connecting a point on p_1 to a point on p_2 is

$$\vec{v} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}.$$

A normal vector of the plane is perpendicular to both vectors $\vec{d}$ and $\vec{v}$:

$$\vec{n} = \vec{d} \times \vec{v}.$$

Compute the cross product:

$$\vec{n} = \begin{pmatrix} 1 \cdot (-5) - 1 \cdot (-3) \\ -\big((-3)(-5) - 1 \cdot 1\big) \\ (-3)(-3) - 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} -5 + 3 \\ -(15 - 1) \\ 9 - 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -14 \\ 8 \end{pmatrix}.$$

We can simplify by dividing by -2:

$$\vec{n} = \begin{pmatrix} 1 \\ 7 \\ -4 \end{pmatrix}.$$

Equation of the plane through a point on p_1, say $(-2, 1, 6)$:

$$\vec{n} \cdot (\vec{x} - \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}) = 0$$

$$1(x + 2) + 7(y - 1) - 4(z - 6) = 0$$

$$x + 2 + 7y - 7 - 4z + 24 = 0 \quad \Rightarrow \quad x + 7y - 4z + 19 = 0$$

Hence, the Cartesian equation of the plane is

$$\boxed{x + 7y - 4z + 19 = 0}.$$

Solution. **Exercise 20.**

a) Plane through points A, B, C

Compute vectors in the plane:

$$\vec{AB} = B - A = \begin{pmatrix} 2 - 0 \\ -1 - 5 \\ -1 - 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ -4 \end{pmatrix}, \quad \vec{AC} = C - A = \begin{pmatrix} -1 - 0 \\ 6 - 5 \\ 5 - 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

Normal vector:

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{pmatrix} (-6) \cdot 2 - (-4) \cdot 1 \\ -(2 \cdot 2 - (-4)(-1)) \\ 2 \cdot 1 - (-6)(-1) \end{pmatrix} = \begin{pmatrix} -12 + 4 \\ -(4 - 4) \\ 2 - 6 \end{pmatrix} = \begin{pmatrix} -8 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \text{ (after dividing by -4).}$$

Equation of the plane through $A(0, 5, 3)$:

$$2(x - 0) + 0(y - 5) + 1(z - 3) = 0 \quad \Rightarrow \quad 2x + z - 3 = 0$$

b) Plane through point P and line g

Direction vector of the line:

$$\vec{d} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$

Vector from P to a point on g, e.g., $Q = (0, -6, -2)$:

$$\vec{v} = Q - P = \begin{pmatrix} 0-4 \\ -6-(-1) \\ -2-1 \end{pmatrix} = \begin{pmatrix} -4 \\ -5 \\ -3 \end{pmatrix}$$

Normal vector to the plane:

$$\vec{n} = \vec{d} \times \vec{v} = \begin{pmatrix} 5\cdot(-3) - 3\cdot(-5) \\ -(1\cdot(-3) - 3\cdot(-4)) \\ 1\cdot(-5) - 5\cdot(-4) \end{pmatrix} = \begin{pmatrix} -15+15 \\ -(-3+12) \\ -5+20 \end{pmatrix} = \begin{pmatrix} 0 \\ -9 \\ 15 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 5 \end{pmatrix} \text{ (divide by 3).}$$

Equation of the plane through $P(4, -1, 1)$:

$$0(x-4) - 3(y+1) + 5(z-1) = 0 \quad \Rightarrow \quad -3y - 3 + 5z - 5 = 0$$

$$-3y + 5z - 8 = 0 \quad \Rightarrow \quad 3y - 5z + 8 = 0$$

Hence, the Cartesian equations are:

$$\text{a) } 2x + z - 3 = 0, \quad \text{b) } 3y - 5z + 8 = 0$$

Solution. **Exercise 21.**

a) Plane containing lines l_1 and l_2

Both lines have the same direction vector:

$$\vec{d} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}.$$

A vector connecting a point on l_1 to a point on l_2 is

$$\vec{v} = \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix}.$$

A normal vector to the plane is

$$\vec{n} = \vec{d} \times \vec{v}.$$

Compute the cross product:

$$\vec{n} = \begin{pmatrix} 2\cdot 2 - 3\cdot(-4) \\ -(0\cdot 2 - 3\cdot 0) \\ 0\cdot(-4) - 2\cdot 0 \end{pmatrix} = \begin{pmatrix} 4+12 \\ -(0-0) \\ 0-0 \end{pmatrix} = \begin{pmatrix} 16 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ after scaling.}$$

Equation of the plane through $P_1 = (5, 1, 2)$:

$$\vec{n} \cdot (\vec{x} - P_1) = 0 \quad \Rightarrow \quad 1(x-5) + 0(y-1) + 0(z-2) = 0$$

$$x - 5 = 0$$

b) Plane through $P(4, -2.5, 1)$ parallel to the xz-plane

A plane parallel to the xz-plane has a normal vector in the y-direction:

$$\vec{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Equation of the plane through P:

$$\vec{n} \cdot (\vec{x} - P) = 0 \quad \Rightarrow \quad 0(x-4) + 1(y+2.5) + 0(z-1) = 0$$

$$y + 2.5 = 0 \quad \Rightarrow \quad y = -2.5$$

Hence, the Cartesian equations are:

$$\text{a) } x = 5, \quad \text{b) } y = -2.5$$

Solution. **Exercise 22.**

A plane in Cartesian form

$$ax + by + cz + d = 0$$

has normal vector

$$\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

The position of the plane can be determined from the components of $\vec{n}$.

a) $\mathbb{P} : 3x - 7y + 3 = 0$

The normal vector is

$$\vec{n} = \begin{pmatrix} 3 \\ -7 \\ 0 \end{pmatrix}.$$

Since the z-component of $\vec{n}$ is zero, the plane is parallel to the z-axis. Moreover, the equation does not contain the variable z, so the plane is parallel to the z-axis and intersects the xy-plane in the line

$$3x - 7y + 3 = 0.$$

b) $\mathbb{P} : 2y - 5z + 9 = 0$

The normal vector is

$$\vec{n} = \begin{pmatrix} 0 \\ 2 \\ -5 \end{pmatrix}.$$

Since the x-component of $\vec{n}$ is zero, the plane is parallel to the x-axis. Moreover, the equation does not contain the variable x, so the plane is parallel to the x-axis and intersects the yz-plane in the line

$$2y - 5z + 9 = 0.$$

c) $\mathbb{P} : 4x - 5 = 0$

The normal vector is

$$\vec{n} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, the plane is perpendicular to the x-axis and parallel to the yz-plane. From the equation,

$$4x - 5 = 0 \quad \Rightarrow \quad x = \frac{5}{4},$$

so the plane is a plane parallel to the yz-plane at distance $\frac{5}{4}$ from the origin.

Solution. **Exercise 23.**

a)

The lines intersect if there exist parameters $s, t \in \mathbb{R}$ such that

$$\begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} + s \begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}.$$

This yields the system

$$\begin{cases} 5 + 5s = 6 + 3t, \\ 3 + 7s = 6 + 5t, \\ 1 - s = 4 + t. \end{cases}$$

From the first equation:

$$5s - 3t = 1.$$

From the second equation:

$$7s - 5t = 3.$$

Solving the system:

$$5s - 3t = 1,$$
$$7s - 5t = 3.$$

Multiply the first equation by 5 and the second by 3:

$$25s - 15t = 5, \qquad 21s - 15t = 9.$$

Subtracting:

$$4s = -4 \quad \Rightarrow \quad s = -1.$$

Then

$$5(-1) - 3t = 1 \quad \Rightarrow \quad -5 - 3t = 1 \quad \Rightarrow \quad t = -2.$$

Thus, the intersection point is

$$P = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix}.$$

A direction vector of the plane is

$$\vec{u} = \begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}.$$

A normal vector is given by the cross product:

$$\vec{n} = \vec{u} \times \vec{v} = \begin{pmatrix} 7 \cdot 1 - (-1) \cdot 5 \\ (-1) \cdot 3 - 5 \cdot 1 \\ 5 \cdot 5 - 7 \cdot 3 \end{pmatrix} = \begin{pmatrix} 12 \\ -8 \\ 4 \end{pmatrix}.$$

We simplify:

$$\vec{n} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}.$$

The plane equation is

$$3(x-0) - 2(y+4) + 1(z-2) = 0.$$

Simplifying:

$$3x - 2y + z - 10 = 0.$$

b)

We solve

$$\begin{pmatrix} 5 \\ 0 \\ 8 \end{pmatrix} + s \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -8 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}.$$

This gives

$$\begin{cases} 5 + 3s = 2, \\ -s = -2 + t, \\ 8 + 4s = -8 + 4t. \end{cases}$$

From the first equation:

$$3s = -3 \quad \Rightarrow \quad s = -1.$$

From the second equation:

$$1 = -2 + t \quad \Rightarrow \quad t = 3.$$

The third equation is satisfied automatically:

$$8 + 4(-1) = 4, \qquad -8 + 4(3) = 4.$$

Thus, the intersection point is

$$P = \begin{pmatrix} 5 \\ 0 \\ 8 \end{pmatrix} - 1 \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}.$$

Direction vectors:

$$\vec{u} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}.$$

Normal vector:

$$\vec{n} = \vec{u} \times \vec{v} = \begin{pmatrix} (-1) \cdot 4 - 4 \cdot 1 \\ 4 \cdot 0 - 3 \cdot 4 \\ 3 \cdot 1 - (-1) \cdot 0 \end{pmatrix} = \begin{pmatrix} -8 \\ -12 \\ 3 \end{pmatrix}.$$

The plane equation is

$$-8(x-2) - 12(y-1) + 3(z-4) = 0.$$

Expanding:

$$-8x - 12y + 3z + 16 + 12 - 12 = 0,$$

$$-8x - 12y + 3z + 16 = 0.$$

Solution. **Exercise 24.**

a)

To find the intercepts with the coordinate axes, we set two coordinates equal to zero.

x-axis: $y = 0,\ z = 0$

$$3x - 6 = 0 \quad \Rightarrow \quad x = 2.$$

Thus, the intercept is $A(2, 0, 0)$.

y-axis: $x = 0,\ z = 0$

$$-y - 6 = 0 \quad \Rightarrow \quad y = -6.$$

Thus, the intercept is $B(0, -6, 0)$.

z-axis: $x = 0,\ y = 0$

$$2z - 6 = 0 \quad \Rightarrow \quad z = 3.$$

Thus, the intercept is $C(0, 0, 3)$.

Therefore, the intercepts are:

$$A(2, 0, 0), \quad B(0, -6, 0), \quad C(0, 0, 3).$$

b)

x-axis: $y = 0,\ z = 0$

$$3x + 36 = 0 \quad \Rightarrow \quad x = -12.$$

Thus, $A(-12, 0, 0)$.

y-axis: $x = 0,\ z = 0$

$$9y + 36 = 0 \quad \Rightarrow \quad y = -4.$$

Thus, $B(0, -4, 0)$.

z-axis: $x = 0,\ y = 0$

$$-3z + 36 = 0 \quad \Rightarrow \quad z = 12.$$

Thus, $C(0, 0, 12)$.

Therefore, the intercepts are:

$$A(-12, 0, 0), \quad B(0, -4, 0), \quad C(0, 0, 12).$$

c)

x-axis: $y = 0,\ z = 0$

$$3x + 60 = 0 \quad \Rightarrow \quad x = -20.$$

Thus, $A(-20, 0, 0)$.

y-axis: $x = 0,\ z = 0$

$$60 = 0,$$

which is impossible. Hence, there is no intercept with the y-axis.

z-axis: $x = 0,\ y = 0$

$$-5z + 60 = 0 \quad \Rightarrow \quad z = 12.$$

Thus, $C(0, 0, 12)$.

Therefore, the intercepts are:

$$A(-20, 0, 0), \quad \text{no intercept on the } y\text{-axis}, \quad C(0, 0, 12).$$

Solution. **Exercise 25.**

The plane intersects the coordinate axes at the points

$$A(a,0,0), \quad B(0,b,0), \quad C(0,0,c).$$

We look for a plane equation of the form

$$\alpha x + \beta y + \gamma z = 1.$$

Since the plane passes through $A(a,0,0)$, we have

$$\alpha a = 1 \quad \Rightarrow \quad \alpha = \frac{1}{a}.$$

Since the plane passes through $B(0,b,0)$, we have

$$\beta b = 1 \quad \Rightarrow \quad \beta = \frac{1}{b}.$$

Since the plane passes through $C(0,0,c)$, we have

$$\gamma c = 1 \quad \Rightarrow \quad \gamma = \frac{1}{c}.$$

Thus, the Cartesian equation of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Multiplying both sides by abc, we obtain

$$bc\,x + ac\,y + ab\,z = abc.$$

This is the desired Cartesian equation after division by abc.

Solution. **Exercise 26.**

We use the intercept form of a plane:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

a)

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{3} = 1.$$

Multiplying by 6, we obtain

$$3x + 2y + 2z - 6 = 0.$$

b)

$$\frac{x}{8} + \frac{y}{-6} + \frac{z}{1} = 1 \quad \Rightarrow \quad \frac{x}{8} - \frac{y}{6} + z = 1.$$

Multiplying by 24, we obtain

$$3x - 4y + 24z - 24 = 0.$$

c)

$$\frac{x}{-9} + \frac{y}{5} + \frac{z}{3} = 1 \quad \Rightarrow \quad -\frac{x}{9} + \frac{y}{5} + \frac{z}{3} = 1.$$

Multiplying by 45, we obtain

$$-5x + 9y + 15z - 45 = 0.$$

d)

$$\frac{x}{-14}+\frac{y}{-14}+\frac{z}{6}=1 \quad \Rightarrow \quad -\frac{x}{14}-\frac{y}{14}+\frac{z}{6}=1.$$

Multiplying by 42, we obtain

$$-3x-3y+7z-42=0.$$

e)

If the plane has only the x-intercept $a=7$, then it is parallel to the yz-plane. Hence, its equation is

$$x=7,$$

or equivalently

$$x-7=0.$$

Solution. **Exercise 27.**

a)

A point lies in the plane if its coordinates satisfy the plane equation.

For $A(5,-2,6)$:

$$3\cdot 5-2\cdot(-2)-5\cdot 6+11=15+4-30+11=0.$$

Thus, $A\in\mathbb{P}$.

For $B(4,6,2)$:

$$3\cdot 4-2\cdot 6-5\cdot 2+11=12-12-10+11=1\neq 0.$$

Thus, $B\notin\mathbb{P}$.

b)

A point lies in the plane if there exist $u,v\in\mathbb{R}$ such that it satisfies the parametric equation.

We compute a normal vector of the plane using the direction vectors:

$$\vec{n}=\begin{pmatrix}-1\\2\\2\end{pmatrix}\times\begin{pmatrix}4\\-4\\-3\end{pmatrix}=\begin{pmatrix}2\\5\\-4\end{pmatrix}.$$

Thus, a Cartesian equation of the plane is

$$2(x-1)+5(y-2)-4(z-2)=0,$$

that is,

$$2x+5y-4z-4=0.$$

For $A(2,4,1)$:

$$2\cdot 2+5\cdot 4-4\cdot 1-4=4+20-4-4=16\neq 0.$$

Thus, $A\notin\mathbb{P}$.

For $B(4,6,2)$:

$$2\cdot 4+5\cdot 6-4\cdot 2-4=8+30-8-4=26\neq 0.$$

Thus, $B\notin\mathbb{P}$.

Solution. **Exercise 28.**

Four points are vertices of a quadrilateral if and only if they are not coplanar. Thus, we check whether the points A, B, C, D lie in one plane.

We compute the vectors

$$\overrightarrow{AB} = \begin{pmatrix} 0-2 \\ 4-1 \\ -3-10 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -13 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} -1-2 \\ 0-1 \\ 5-10 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ -5 \end{pmatrix}, \quad \overrightarrow{AD} = \begin{pmatrix} 3-2 \\ 5-1 \\ 2-10 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -8 \end{pmatrix}.$$

The points are coplanar if and only if the scalar triple product is zero:

$$\det \begin{pmatrix} -2 & 3 & -13 \\ -3 & -1 & -5 \\ 1 & 4 & -8 \end{pmatrix} = 0.$$

We compute the determinant:

$$\det \begin{pmatrix} -2 & 3 & -13 \\ -3 & -1 & -5 \\ 1 & 4 & -8 \end{pmatrix} = -2 \begin{vmatrix} -1 & -5 \\ 4 & -8 \end{vmatrix} - 3 \begin{vmatrix} -3 & -5 \\ 1 & -8 \end{vmatrix} - 13 \begin{vmatrix} -3 & -1 \\ 1 & 4 \end{vmatrix}.$$

$$= -2(8+20) - 3(24+5) - 13(-12+1) = -2 \cdot 28 - 3 \cdot 29 - 13(-11).$$

$$= -56 - 87 + 143 = 0.$$

Since the determinant is zero, the points A, B, C, D lie in one plane. Therefore, they do not form a quadrilateral in space.

Hence, the points are not vertices of a spatial quadrilateral.

Solution. **Exercise 29.**

The intersection point satisfies the system

$$\begin{cases} 3x - 2y + z = 8, \\ x + y - 3z = -4, \\ 4x + 3y - 5z = 8. \end{cases}$$

From the second equation,

$$x = -4 - y + 3z.$$

Substituting into the first equation:

$$3(-4 - y + 3z) - 2y + z = 8,$$

$$-12 - 3y + 9z - 2y + z = 8,$$

$$-12 - 5y + 10z = 8,$$

$$-5y + 10z = 20,$$

$$y - 2z = -4.$$

Substituting into the third equation:

$$4(-4 - y + 3z) + 3y - 5z = 8,$$

$$-16 - 4y + 12z + 3y - 5z = 8,$$

$$-16 - y + 7z = 8,$$

$$-y + 7z = 24.$$

Now solve

$$\begin{cases} y - 2z = -4, \\ -y + 7z = 24. \end{cases}$$

Adding both equations:

$$5z = 20 \quad \Rightarrow \quad z = 4.$$

Then

$$y - 2 \cdot 4 = -4 \quad \Rightarrow \quad y = 4.$$

Finally,

$$x = -4 - y + 3z = -4 - 4 + 12 = 4.$$

Therefore, the intersection point is

$$(4, 4, 4).$$

Solution. **Exercise 30.**

The intercepts of the plane with the coordinate axes are obtained by setting two coordinates equal to zero.

For the x-axis ($y = 0, z = 0$):

$$9x - 144 = 0 \quad \Rightarrow \quad x = 16.$$

Thus,

$$A(16, 0, 0).$$

For the y-axis ($x = 0, z = 0$):

$$16y - 144 = 0 \quad \Rightarrow \quad y = 9.$$

Thus,

$$B(0, 9, 0).$$

For the z-axis ($x = 0, y = 0$):

$$cz - 144 = 0 \quad \Rightarrow \quad z = \frac{144}{c}.$$

Thus,

$$C\left(0, 0, \frac{144}{c}\right).$$

The volume of the pyramid (tetrahedron) with vertices O, A, B, C is

$$V = \frac{1}{6} \left| \det\left(\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}\right) \right|.$$

We have

$$\overrightarrow{OA} = \begin{pmatrix} 16 \\ 0 \\ 0 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 0 \\ 9 \\ 0 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 0 \\ 0 \\ \frac{144}{c} \end{pmatrix}.$$

Thus,

$$\det\left(\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}\right) = 16 \cdot 9 \cdot \frac{144}{c} = \frac{20736}{c}.$$

Therefore,

$$V = \frac{1}{6} \left| \frac{20736}{c} \right|.$$

Given that $V = 384$, we obtain

$$\frac{1}{6} \cdot \frac{20736}{|c|} = 384,$$

$$\frac{20736}{|c|} = 2304,$$

$$|c| = 9.$$

Hence,

$$c = \pm 9.$$

Solution. **Exercise 31.**

a)

The line intersects the plane if there exists t such that the parametric coordinates satisfy the plane equation.

Parametrize the line:

$$x = -4 + 2t, \quad y = -10 + 3t, \quad z = 8 - 2t.$$

Substitute into the plane equation:

$$3(-4 + 2t) - (-10 + 3t) + 4(8 - 2t) - 15 = 0.$$

Compute step by step:

$$-12 + 6t + 10 - 3t + 32 - 8t - 15 = 0,$$

$$(6t - 3t - 8t) + (-12 + 10 + 32 - 15) = 0,$$

$$-5t + 15 = 0 \quad \Rightarrow \quad t = 3.$$

Hence, the intersection point is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -10 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.$$

b)

Parametrize the line:

$$x = 7 - 2t, \quad y = 2 + 3t, \quad z = 9 + 4t.$$

Substitute into the plane equation:

$$4(2 + 3t) - 3(9 + 4t) + 27 = 0.$$

Compute step by step:

$$8 + 12t - 27 - 12t + 27 = 0,$$

$$8 - 27 + 27 + (12t - 12t) = 0,$$

$$8 = 0.$$

This is impossible. Therefore, the line does **not intersect** the plane.

Solution. **Exercise 32.**

The foot of the altitude D from A onto side BC lies on the line through B and C:

$$\overrightarrow{BC} = \begin{pmatrix} 5-(-7) \\ -3-3 \\ 7-(-5) \end{pmatrix} = \begin{pmatrix} 12 \\ -6 \\ 12 \end{pmatrix}.$$

Parametrize BC:

$$\overrightarrow{BD} = t\overrightarrow{BC} \quad \Rightarrow \quad D = B + t\overrightarrow{BC} = \begin{pmatrix} -7 \\ 3 \\ -5 \end{pmatrix} + t \begin{pmatrix} 12 \\ -6 \\ 12 \end{pmatrix}.$$

Since $AD \perp BC$, the vector $\overrightarrow{AD} = D - A$ is orthogonal to $\overrightarrow{BC}$:

$$\overrightarrow{AD} \cdot \overrightarrow{BC} = 0.$$

Compute $\overrightarrow{AD}$:

$$\overrightarrow{AD} = D - A = \begin{pmatrix} -7+12t-0 \\ 3-6t-1 \\ -5+12t-5 \end{pmatrix} = \begin{pmatrix} -7+12t \\ 2-6t \\ -10+12t \end{pmatrix}.$$

Dot product condition:

$$(-7+12t, 2-6t, -10+12t) \cdot (12, -6, 12) = 0.$$

Compute:

$$(-7+12t) \cdot 12 + (2-6t)(-6) + (-10+12t) \cdot 12 = 0,$$
$$-84 + 144t - 12 + 36t - 120 + 144t = 0,$$
$$324t - 216 = 0 \quad \Rightarrow \quad t = \frac{216}{324} = \frac{2}{3}.$$

Hence, the coordinates of D are:

$$D = B + \frac{2}{3}\overrightarrow{BC} = \begin{pmatrix} -7 \\ 3 \\ -5 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 12 \\ -6 \\ 12 \end{pmatrix} = \begin{pmatrix} -7+8 \\ 3-4 \\ -5+8 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}.$$

Area of triangle ABC

The area of a triangle is

$$\text{Area} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|.$$

Compute

$$\overrightarrow{AB} = B - A = \begin{pmatrix} -7 \\ 3 \\ -5 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -7 \\ 2 \\ -10 \end{pmatrix},$$

$$\overrightarrow{AC} = C - A = \begin{pmatrix} 5 \\ -3 \\ 7 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}.$$

Compute the cross product:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} (2)(2) - (-10)(-4) \\ -((-7)(2) - (-10)(5)) \\ (-7)(-4) - (2)(5) \end{pmatrix} = \begin{pmatrix} 4 - 40 \\ -(-14+50) \\ 28 - 10 \end{pmatrix} = \begin{pmatrix} -36 \\ -36 \\ 18 \end{pmatrix}.$$

Magnitude:

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-36)^2 + (-36)^2 + 18^2} = \sqrt{1296 + 1296 + 324} = \sqrt{2916} = 54.$$

Hence, the area is

$$\text{Area} = \frac{1}{2} \cdot 54 = 27.$$

Answer:

$$D = (1, -1, 3), \quad \text{Area of } \triangle ABC = 27.$$

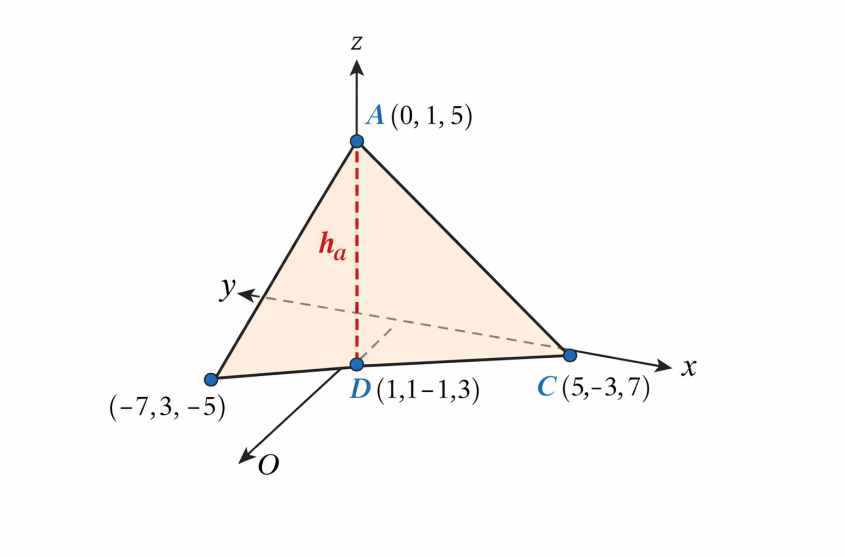


Figure 2: Triangle ABC with vertices $A(0, 1, 5)$, $B(-7, 3, -5)$, $C(5, -3, 7)$ and altitude AD. Point D is the foot of the altitude from A onto side BC.

Solution. **Exercise 33.**

Let the plane we want be $\mathbb{Q}$. Since $\mathbb{Q}$ is **normal to** $\mathcal{P}$, its normal vector $\vec{n}_{\mathbb{Q}}$ is *parallel to the normal vector of* $\mathcal{P}$.

The normal vector of $\mathcal{P}$ is

$$\vec{n}_{\mathcal{P}} = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}.$$

Hence, we can write the cartesian equation of $\mathbb{Q}$ in the form:

$$3(x - x_0) + 1(y - y_0) + 7(z - z_0) = 0,$$

where (x_0, y_0, z_0) is any point on the plane $\mathbb{Q}$.

We know $\mathbb{Q}$ contains $P(2, 0, 1)$, so we can plug it in:

$$3(x - 2) + 1(y - 0) + 7(z - 1) = 0.$$

Simplify:

$$3x - 6 + y + 7z - 7 = 0 \quad \Rightarrow \quad 3x + y + 7z - 13 = 0.$$

Verification: The plane also contains $Q(-3, -4, 2)$:

$$3(-3) + (-4) + 7(2) - 13 = -9 - 4 + 14 - 13 = -12 \neq 0.$$

Ah — careful! This shows a subtlety: a plane normal to $\mathcal{P}$ is not necessarily **parallel** to $\vec{n}_{\mathcal{P}}$; rather, its **normal vector must be orthogonal** to $\vec{n}_{\mathcal{P}}$.

—

Correct approach:

Let the plane $\mathbb{Q}$ have normal vector $\vec{n}_{\mathbb{Q}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Since $\mathbb{Q}$ is normal to $\mathcal{P}$:

$$\vec{n}_{\mathbb{Q}} \cdot \vec{n}_{\mathcal{P}} = 0 \quad \Rightarrow \quad 3a + 1 \cdot b + 7c = 0.$$

Also, $\mathbb{Q}$ contains points P and Q, so the vector $\overrightarrow{PQ}$ lies in the plane:

$$\overrightarrow{PQ} = Q - P = \begin{pmatrix} -3 - 2 \\ -4 - 0 \\ 2 - 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 1 \end{pmatrix}.$$

Since $\overrightarrow{PQ}$ lies in the plane:

$$\vec{n}_{\mathbb{Q}} \cdot \overrightarrow{PQ} = 0 \quad \Rightarrow \quad -5a - 4b + c = 0.$$

Now we have two equations for (a, b, c):

$$\begin{cases} 3a + b + 7c = 0 \\ -5a - 4b + c = 0 \end{cases}$$

Solve for a simple integer solution. Let's pick $c = 1$:

$$3a + b + 7 = 0 \quad \Rightarrow \quad b = -3a - 7$$

$$-5a - 4(-3a - 7) + 1 = 0 \quad \Rightarrow \quad -5a + 12a + 28 + 1 = 0 \quad \Rightarrow \quad 7a + 29 = 0 \quad \Rightarrow \quad a = -\frac{29}{7}.$$

Then

$$b = -3(-29/7) - 7 = 87/7 - 49/7 = 38/7, \quad c = 1.$$

Multiply all coefficients by 7 to avoid fractions:

$$a = -29, \quad b = 38, \quad c = 7.$$

Hence, a normal vector of $\mathbb{Q}$ is $\vec{n}_{\mathbb{Q}} = (-29, 38, 7)$.
The plane passes through $P(2, 0, 1)$:

$$-29(x - 2) + 38(y - 0) + 7(z - 1) = 0,$$

$$-29x + 58 + 38y + 7z - 7 = 0,$$

$$-29x + 38y + 7z + 51 = 0 \quad \Rightarrow \quad 29x - 38y - 7z - 51 = 0.$$

Answer:

$$\boxed{29x - 38y - 7z - 51 = 0}.$$

Solution. **Exercise 34.**

Let $P' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ be the reflection of P in the plane $\mathcal{P}$.

The formula for the reflection of a point P across a plane with equation

$$ax + by + cz + d = 0$$

is

$$P' = P - 2\frac{ax_P + by_P + cz_P + d}{a^2 + b^2 + c^2} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Here, $a = 4, b = -3, c = 1, d = 4$ and $P = (0, 4, -5)$.

Step 1: Compute the signed distance factor:

$$ax_P + by_P + cz_P + d = 4 \cdot 0 + (-3) \cdot 4 + 1 \cdot (-5) + 4 = 0 - 12 - 5 + 4 = -13.$$

Step 2: Compute $a^2 + b^2 + c^2$:

$$4^2 + (-3)^2 + 1^2 = 16 + 9 + 1 = 26.$$

Step 3: Multiply the normal vector by $2\frac{-13}{26} = -1$:

$$-1 \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix}.$$

Step 4: Add this vector to the original point P:

$$P' = P - 2\frac{ax_P + by_P + cz_P + d}{a^2 + b^2 + c^2} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ -5 \end{pmatrix} + \begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \\ -6 \end{pmatrix}.$$

Answer:

$$\boxed{P' = \begin{pmatrix} -4 \\ 7 \\ -6 \end{pmatrix}}.$$

Solution. **Exercise 35.**

Step 1: Foot of perpendicular.

Let H be the foot of the perpendicular from P to the line ℓ. By definition of reflection, H is the midpoint of PP', so

$$P' = P + 2\overrightarrow{PH}.$$

Step 2: Express $\overrightarrow{PH}$ along the line.

The vector from a point on the line to the foot H can be written as:

$$\overrightarrow{AH} = \lambda\vec{v}, \quad \lambda \in \mathbb{R}.$$

Then

$$H = A + \lambda\vec{v}.$$

Step 3: Use perpendicularity condition.

The vector $\overrightarrow{PH}$ is perpendicular to the line direction:

$$\overrightarrow{PH} \cdot \vec{v} = 0 \quad \Rightarrow \quad (H - P) \cdot \vec{v} = 0.$$

Substitute $H = A + \lambda\vec{v}$:

$$(A + \lambda\vec{v} - P) \cdot \vec{v} = 0.$$

Solve for λ:

$$(A - P) \cdot \vec{v} + \lambda\vec{v} \cdot \vec{v} = 0 \quad \Rightarrow \quad \lambda = \frac{(P - A) \cdot \vec{v}}{|\vec{v}|^2}.$$

Step 4: Express the foot H and reflected point P'.

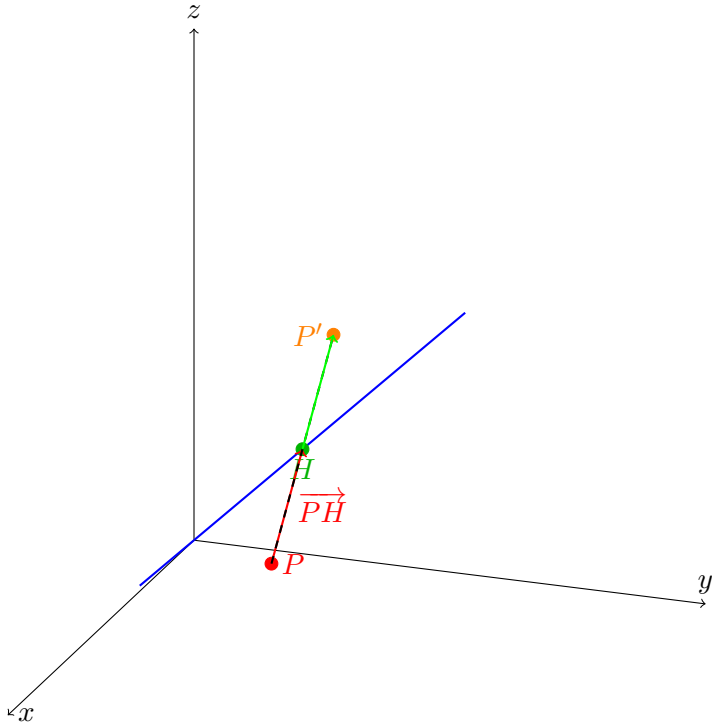
$$H = A + \lambda\vec{v} = A + \frac{(P - A) \cdot \vec{v}}{|\vec{v}|^2}\vec{v},$$

$$P' = P + 2(H - P) = 2H - P = 2\left(A + \frac{(P - A) \cdot \vec{v}}{|\vec{v}|^2}\vec{v}\right) - P.$$

Step 5: Summary formula.

$$\boxed{P' = 2\left(A + \frac{(P - A) \cdot \vec{v}}{|\vec{v}|^2}\vec{v}\right) - P}$$

Step 6: Visualization.



Solution. **Exercise 36.**
Derivation of the reflection formula:

Step 1: Foot of perpendicular.

Let H be the foot of the perpendicular from P to the plane $\mathcal{P} : ax + by + cz + d = 0$. By definition of reflection, H is the midpoint of PP', so

$$P' = P + 2\overrightarrow{PH}.$$

Step 2: Vector along plane normal.

The plane's normal is

$$\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad \overrightarrow{PH} = \lambda\vec{n}.$$

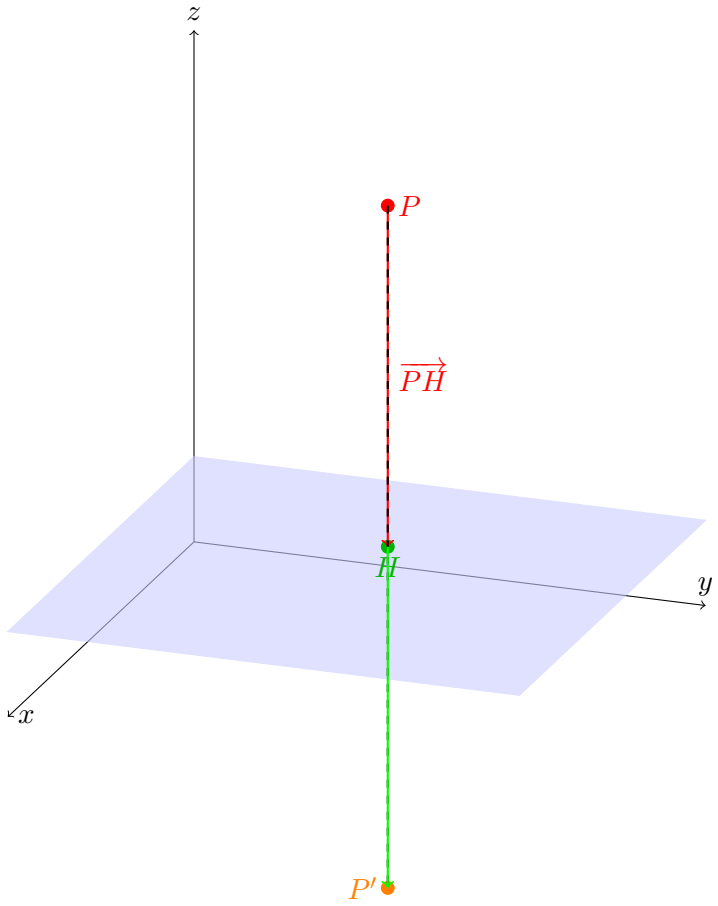
Step 3: Solve for λ.
H lies on the plane:

$$a(x_P + \lambda a) + b(y_P + \lambda b) + c(z_P + \lambda c) + d = 0 \quad \Rightarrow \quad \lambda = -\frac{ax_P + by_P + cz_P + d}{a^2 + b^2 + c^2}.$$

Step 4: Reflected point.

$$P' = P + 2\lambda\vec{n} = P - 2\frac{ax_P + by_P + cz_P + d}{a^2 + b^2 + c^2}\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Step 5: 3D Visualization.



Solution. **Exercise 37.**
Step 1: Midpoint of P and P'.
Let H be the foot of the perpendicular from P to the plane $\mathcal{P}$. By definition of reflection, H lies exactly midway between P and P':

$$H = \frac{P + P'}{2} = \frac{(4, 3, -2) + (0, 0, 7)}{2} = \frac{(4, 3, 5)}{2} = (2, 1.5, 2.5).$$

Step 2: Normal vector of the plane.
The vector $\overrightarrow{PP'} = P' - P$ is perpendicular to the plane:

$$\overrightarrow{PP'} = (0 - 4, 0 - 3, 7 - (-2)) = (-4, -3, 9).$$

Hence, the plane has normal vector

$$\vec{n} = (-4, -3, 9).$$

Step 3: Cartesian equation of the plane.
A plane with normal vector $\vec{n} = (A, B, C)$ passing through $H = (x_0, y_0, z_0)$ has equation

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Substitute $\vec{n} = (-4, -3, 9)$ and $H = (2, 1.5, 2.5)$:

$$-4(x - 2) - 3(y - 1.5) + 9(z - 2.5) = 0.$$

Expanding:

$$-4x + 8 - 3y + 4.5 + 9z - 22.5 = 0 \quad \Rightarrow \quad -4x - 3y + 9z - 10 = 0.$$

Multiply by -1 for nicer form:

$$\boxed{4x + 3y - 9z + 10 = 0}.$$

This is the Cartesian equation of the plane $\mathcal{P}$.

Solution. **Exercise 38.**
Step 1: Identify line points and direction.
The line $\updownarrow$ passes through

$$A = (-1, -1, 1)$$

with direction vector

$$\vec{v} = (-3, 4, 3).$$

The plane has equation

$$\mathcal{P} : x - 3y - 2z + 42 = 0$$

so its normal vector is

$$\vec{n} = (1, -3, -2).$$

—

Step 2: Reflect the point A across the plane.
The reflection A' of point A is given by

$$\overrightarrow{AA'} = 2\overrightarrow{AH} = 2\frac{(A \cdot \vec{n} + D)}{\|\vec{n}\|^2}\vec{n},$$

where H is the foot of the perpendicular from A to $\mathcal{P}$ and $D = 42$ is from the plane equation $n \cdot X + D = 0$.

Compute $A \cdot \vec{n}$:

$$A \cdot \vec{n} = (-1) \cdot 1 + (-1) \cdot (-3) + 1 \cdot (-2) = -1 + 3 - 2 = 0.$$

Hence,

$$\overrightarrow{AA'} = 2\frac{0 + 42}{1^2 + (-3)^2 + (-2)^2}(1, -3, -2) = 2\frac{42}{1 + 9 + 4}(1, -3, -2) = 2\frac{42}{14}(1, -3, -2) = 6(1, -3, -2).$$

Therefore,

$$A' = A + \overrightarrow{AA'} = (-1, -1, 1) + 6(1, -3, -2) = (-1 + 6, -1 - 18, 1 - 12) = (5, -19, -11).$$

—

Step 3: Reflect the direction vector.
Let $\vec{v}_\parallel$ be the component of $\vec{v}$ parallel to the plane, and $\vec{v}_\perp$ perpendicular:

$$\vec{v}_\perp = \frac{\vec{v} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n}, \quad \vec{v}_\parallel = \vec{v} - \vec{v}_\perp.$$

Compute $\vec{v} \cdot \vec{n}$:

$$\vec{v} \cdot \vec{n} = (-3) \cdot 1 + 4 \cdot (-3) + 3 \cdot (-2) = -3 - 12 - 6 = -21.$$

$$\vec{v}_\perp = \frac{-21}{14}(1, -3, -2) = \left(-\frac{21}{14}, \frac{63}{14}, \frac{42}{14}\right) = \left(-\frac{3}{2}, \frac{9}{2}, 3\right).$$

$$\vec{v}_\parallel = \vec{v} - \vec{v}_\perp = \left(-3 + \frac{3}{2}, 4 - \frac{9}{2}, 3 - 3\right) = \left(-\frac{3}{2}, -\frac{1}{2}, 0\right).$$

The reflected vector is

$$\vec{v}' = \vec{v}_\parallel - \vec{v}_\perp = \left(-\frac{3}{2}, -\frac{1}{2}, 0\right) - \left(-\frac{3}{2}, \frac{9}{2}, 3\right) = (0, -5, -3).$$

—

Step 4: Vector equation of the reflected line.
The reflected line $\updownarrow'$ passes through A' with direction $\vec{v}'$:

$$\boxed{\updownarrow' : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -19 \\ -11 \end{pmatrix} + t \begin{pmatrix} 0 \\ -5 \\ -3 \end{pmatrix}, \quad t \in \mathbb{R}}$$

Solution. **Exercise 39.**
Step 1: Parameterize the line of the reflected ray.
The reflected ray passes through Q with direction vector $\vec{v} = \overrightarrow{RQ}$ unknown. But geometrically, the reflection point R lies on both the plane $\mathcal{P}$ and the line connecting R to Q.

We can parameterize the line from Q backward along the ray:

$$\vec{X}(t) = Q + t\vec{d} = (7, -1, 8) + t(x_R - 7, y_R + 1, z_R - 8)$$

but instead, we can use the reflection property: the **incoming vector** $\overrightarrow{PR}$ and the **outgoing vector** $\overrightarrow{RQ}$ satisfy the **law of reflection**:

$$\overrightarrow{RQ} - \overrightarrow{PR} \parallel \vec{n},$$

where $\vec{n} = (5, -2, 3)$ is the normal vector of the plane.

—

Step 2: Express the reflection condition.
Let $R = (x, y, z)$ be the point of reflection. Then

$$\overrightarrow{PR} = R - P = (x - 7, y + 7, z - 4), \quad \overrightarrow{RQ} = Q - R = (7 - x, -1 - y, 8 - z).$$

The reflection law states that the **component of $\overrightarrow{PR}$ along the normal is reversed**:

$$\overrightarrow{RQ} - \overrightarrow{PR} = \lambda\vec{n}, \quad \lambda \in \mathbb{R}.$$

Compute the difference:

$$\overrightarrow{RQ} - \overrightarrow{PR} = (7 - x - (x - 7), -1 - y - (y + 7), 8 - z - (z - 4)) = (14 - 2x, -8 - 2y, 4 - 2z).$$

Hence, we require

$$(14 - 2x, -8 - 2y, 4 - 2z) = \lambda(5, -2, 3).$$

—

Step 3: Solve for x, y, z.
Equate components:
1. $14 - 2x = 5\lambda \quad \Rightarrow \quad x = 7 - \frac{5}{2}\lambda$ 2. $-8 - 2y = -2\lambda \quad \Rightarrow \quad y = -4 + \lambda$ 3. $4 - 2z = 3\lambda \quad \Rightarrow \quad z = 2 - \frac{3}{2}\lambda$

—

Step 4: Impose the plane condition.
Since $R \in \mathcal{P}$:

$$5x - 2y + 3z - 23 = 0.$$

Substitute the expressions in terms of λ:

$$5\left(7 - \frac{5}{2}\lambda\right) - 2(-4 + \lambda) + 3\left(2 - \frac{3}{2}\lambda\right) - 23 = 0.$$

Simplify step by step:
1. $5 \cdot 7 - 5 \cdot \frac{5}{2}\lambda = 35 - \frac{25}{2}\lambda$ 2. $-2(-4 + \lambda) = 8 - 2\lambda$ 3. $3(2 - \frac{3}{2}\lambda) = 6 - \frac{9}{2}\lambda$
Sum constants: $35 + 8 + 6 - 23 = 26$ Sum λ terms: $-\frac{25}{2} - 2 - \frac{9}{2} = -\frac{25+4+9}{2} = -\frac{38}{2} = -19$
Hence:

$$26 - 19\lambda = 0 \quad \Rightarrow \quad \lambda = \frac{26}{19}.$$

—

Step 5: Compute R.

$$x = 7 - \frac{5}{2} \cdot \frac{26}{19} = 7 - \frac{130}{38} = 7 - \frac{65}{19} = \frac{133 - 65}{19} = \frac{68}{19}$$

$$y = -4 + \frac{26}{19} = -\frac{76}{19} + \frac{26}{19} = -\frac{50}{19}$$

$$z = 2 - \frac{3}{2} \cdot \frac{26}{19} = 2 - \frac{78}{38} = 2 - \frac{39}{19} = \frac{38 - 39}{19} = -\frac{1}{19}$$

—

Answer: The ray is reflected at

$$\boxed{R = \left(\frac{68}{19}, -\frac{50}{19}, -\frac{1}{19}\right)}.$$

Solution. **Exercise 40.**
a) Distance using the Hessian normal form.

Rewrite the line in standard form:

$$y = 2.4x - 0.4 \quad \Rightarrow \quad 2.4x - y - 0.4 = 0.$$

Here, $A = 2.4$, $B = -1$, $C = -0.4$, and $P = (x_0, y_0) = (3, -1)$.
Compute the distance:

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}} = \frac{|2.4 \cdot 3 + (-1) \cdot (-1) - 0.4|}{\sqrt{2.4^2 + (-1)^2}}.$$

Simplify numerator:

$$2.4 \cdot 3 = 7.2, \quad -1 \cdot (-1) = 1, \quad -0.4 \quad \Rightarrow \quad 7.2 + 1 - 0.4 = 7.8$$

Denominator:

$$\sqrt{2.4^2 + 1} = \sqrt{5.76 + 1} = \sqrt{6.76} \approx 2.6$$

Hence,

$$\boxed{d \approx \frac{7.8}{2.6} = 3}.$$

—

b) Distance from a point to a line in vector form.

The line is given by

$$\updownarrow : (x, y) = \begin{pmatrix} -5 \\ -2 \end{pmatrix} + t \begin{pmatrix} 15 \\ 8 \end{pmatrix}.$$

Let $P = (6, 5)$. The distance from P to the line is

$$d = \frac{\|\overrightarrow{AB} \times \overrightarrow{AP}\|}{\|\overrightarrow{AB}\|},$$

where $\overrightarrow{AB}$ is the direction vector of the line, and $\overrightarrow{AP}$ is the vector from a point on the line to P.

Here, $\overrightarrow{AB} = (15, 8)$, $\overrightarrow{AP} = (6 + 5, 5 + 2) = (11, 7)$.

Compute the 2D "cross product" (determinant):

$$|\overrightarrow{AB} \times \overrightarrow{AP}| = |15 \cdot 7 - 8 \cdot 11| = |105 - 88| = 17$$

Norm of the direction vector:

$$\|\overrightarrow{AB}\| = \sqrt{15^2 + 8^2} = \sqrt{225 + 64} = \sqrt{289} = 17$$

Hence,

$$d = \frac{17}{17} = 1.$$

$$\boxed{d = 1}$$

Solution. **Exercise 41.**

Step 1: Identify points and direction vectors.

Let

$$\vec{r}_1 = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix}, \quad \vec{u}_1 = \begin{pmatrix} 1 \\ 12 \\ 0 \end{pmatrix}, \quad \vec{r}_2 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 12 \\ -1 \end{pmatrix}.$$

—

Step 2: Compute the cross product of direction vectors.

$$\vec{u}_1 \times \vec{u}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 12 & 0 \\ 0 & 12 & -1 \end{vmatrix}.$$

Compute components:

- x-component: $12 \cdot (-1) - 0 \cdot 12 = -12$ - y-component: $-(1 \cdot (-1) - 0 \cdot 0) = -(-1) = 1$ - z-component: $1 \cdot 12 - 12 \cdot 0 = 12$

Hence,

$$\vec{u}_1 \times \vec{u}_2 = \begin{pmatrix} -12 \\ 1 \\ 12 \end{pmatrix}.$$

—

Step 3: Compute the vector connecting points on the lines.

$$\vec{r}_2 - \vec{r}_1 = \begin{pmatrix} 2-3 \\ 4-(-3) \\ 1-4 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -3 \end{pmatrix}.$$

—

Step 4: Compute the scalar triple product.

$$(\vec{r}_2 - \vec{r}_1) \cdot (\vec{u}_1 \times \vec{u}_2) = (-1)(-12) + 7 \cdot 1 + (-3) \cdot 12 = 12 + 7 - 36 = -17.$$

Take absolute value:

$$|(\vec{r}_2 - \vec{r}_1) \cdot (\vec{u}_1 \times \vec{u}_2)| = 17.$$

—

Step 5: Compute norm of the cross product.

$$\|\vec{u}_1 \times \vec{u}_2\| = \sqrt{(-12)^2 + 1^2 + 12^2} = \sqrt{144 + 1 + 144} = \sqrt{289} = 17.$$

—

Step 6: Compute the distance.

$$d(\ell_1, \ell_2) = \frac{17}{17} = 1.$$

$$\boxed{d = 1}$$