<u>*Note:*</u> *these exercises are some of the calculations that everybody should have done at least once in her/his life. At least, I always return to them every once in a while since they keep on popping up.*

<u>Exercise 1:</u> instantaneous, average, and complex power

Consider the synchronous AC voltage and current signals

$$v(t) = |V| \cos(\omega t + \theta_V) \ , \ i(t) = |I| \cos(\omega t + \theta_I) \, .$$

Let $\phi = \theta_V - \theta_I$ be the phase difference between voltage and current, and let $T = 2\pi/\omega$ be the period of the AC signals. Show the following identities:

a) instantaneous power: $p(t) = v(t) \cdot i(t) = \frac{1}{2} \cdot |V| \cdot |I| \cdot \cos(\theta_V - \theta_I) + \frac{1}{2} \cdot |V| \cdot |I| \cdot \cos(2\omega t + \theta_V + \theta_I)$

b) active power: $P = \frac{1}{T} \int_0^T v(t) \cdot i(t) \, dt = \frac{1}{2} \cdot |V| \cdot |I| \cdot \cos(\phi)$

c) reactive power: $Q = \frac{1}{T} \int_0^T v(t) \cdot i(t - T/4) \, dt = \frac{1}{2} \cdot |V| \cdot |I| \cdot \sin(\phi)$

After having worked through the tedious integrals and trigonometric identities required to derive the above identities you will appreciate the following complex power phasor calculus. Consider the normalized phasor variables associated to the AC voltage and current signals:

$$V = \frac{1}{\sqrt{2}} |V| e^{\mathrm{i}\theta_V} \ , \ I = \frac{1}{\sqrt{2}} |I| e^{\mathrm{i}\theta_V} \, .$$

Next, calculate the

d) complex power: $S = V \cdot \bar{I}$

Verify that $S = V \cdot \bar{I} = P + \mathrm{i} \, Q$, where P and Q are as in b) and c) respectively.

<u>Exercise 2:</u> power dissipated by loads

Consider the static impedance loads in Figure 1. Calculate the power factors as well as the active and reactive power dissipated by the three different loads as well as associated power factors.

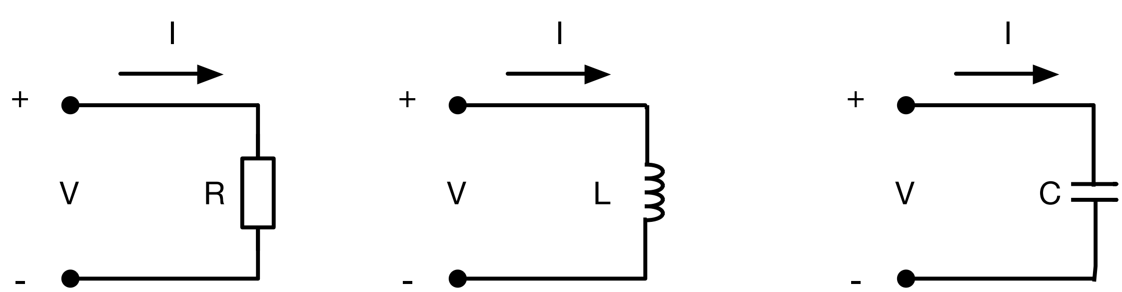

Figure 1: Static impedance loads

Exercise 3: polar representation of power flow equations in lossless network

Consider a lossless and inductive network. Derive the power flow equations in polar form:

$$P_i = \sum\nolimits_{j=1}^{n} B_{ij} E_i E_j \sin(\theta_i - \theta_j)\,,$$
$$Q_i = -\sum\nolimits_{j=1}^{n} B_{ij} E_i E_j \cos(\theta_i - \theta_j)\,.$$

Exercise 4: circuit tricks in a grid with identical lines

Consider a network with constant R/X ratios, that is, all lines in the network are made from the same material and thus have uniform per-unit-length resistance-to-reactance ratios: $z_{ij} = |z_{ij}|e^{\mathrm{i}\varphi}$ for some angle $\varphi \in [-\pi/2, \pi/2]$ for all $\{i, j\} \in \mathcal{E}$.

Derive the polar power flow equations, and show that they take the form

$$\tilde{P}_i = \sum\nolimits_{j=1}^{n} B_{ij} E_i E_j \sin(\theta_i - \theta_j)\,,$$
$$\tilde{Q}_i = -\sum\nolimits_{j=1}^{n} B_{ij} E_i E_j \cos(\theta_i - \theta_j)\,,$$

where the power injections $(\tilde{P}, \tilde{Q})$ are related to the original injections (P, Q) as

$$\begin{bmatrix} \tilde{P}_i \\ \tilde{Q}_i \end{bmatrix} = \begin{bmatrix} \sin(\varphi) & -\cos(\varphi) \\ \cos(\varphi) & \sin(\varphi) \end{bmatrix} \begin{bmatrix} P_i \\ Q_i \end{bmatrix}.$$

<u>Remark:</u> *In the power electronics literature this transformation is often used as an output filter, e.g., when controllers are designed for an inductive network need to be adapted to a resistive network. Of course, now active and reactive power become mixed so that concepts like active power sharing become ill-defined. It is also not misleading when analyzing networks PV and PQ buses.* □

Exercise 5: Coupled oscillators and swing equations

Consider a network of n coupled oscillators as illustrated in Figure 2. Assume that

- the n oscillators live on a ring of unit radius and can pass another without colliding;
- each oscillator $i \in \{1, \ldots, n\}$ is parameterized by its inertial and viscous damping coefficients $M_i > 0$ and $D_i > 0$, respectively; and
- furthermore, each oscillator $i \in \{1, \ldots, n\}$ is subject to a constant exogenous torque $P_i \in \mathbb{R}$ as well as the coupling torque through the perfectly elastic springs with stiffness coefficients k_{ij} that are positive $k_{ij} > 0$ if j is a neighbor of i and zero otherwise.

Your tasks are as follows:

a) Show that the system of coupled oscillators obeys the swing dynamics

$$M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_i - \sum\nolimits_{j=1}^{n} k_{ij} \sin(\theta_i - \theta_j)\,, \quad i \in \{1, \ldots, n\}\,.$$

b) Assume that the above model in a) has a synchronous solution $\dot{\theta}_i = \omega_{\text{sync}} \in \mathbb{R}$ for all $i \in \{1, \ldots, n\}$. Calculate ω_{sync} explicitly as a function of the problem parameters.

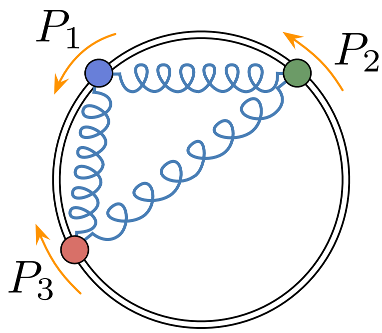

Figure 2: A network of heterogeneous coupled oscillators with elastic coupling

Exercise 6: Energy functions of structure-preserving power system models

Consider a structure-preserving power network model consisting of generator buses G and load buses L so that $G \cup L = \{1, \ldots, n\}$. The generators are modeled by the swing dynamics and the loads demand constant power:

$$i \in G \,:\, M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_i - \sum\nolimits_{j=1}^{n} B_{ij} \sin(\theta_i - \theta_j)\,,$$
$$i \in L \,:\, 0 = P_i - \sum\nolimits_{j=1}^{n} B_{ij} \sin(\theta_i - \theta_j)\,.$$

Your tasks are as follows:

a) Find a so-called energy function $V : \mathbb{T}^{|G|+|L|} \times \mathbb{R}^{|G|} \to \mathbb{R}$, $\left(\theta(t), \dot{\theta}(t)\right) \mapsto V(\theta(t), \dot{\theta}(t))$ (not necessarily positive definite) whose derivative is non-increasing $\dot{V}(\theta(t), \dot{\theta}(t)) \leq 0$;

b) which further analysis steps would you have to undertake to prove local (or global) stability of equilibria of the structure-preserving power network model via the energy function; and

c) extend the arguments from a) to the presence of voltage dynamics and reactive power flows

$$i \in G \,:\, M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_i - \sum\nolimits_{j=1}^{n} B_{ij} E_i E_j \sin(\theta_i - \theta_j)\,,$$
$$i \in G \,:\, T_i \dot{E}_i = -K_i E_i + \sum\nolimits_{j=1}^{n} B_{ij} E_j \cos(\theta_i - \theta_j) + F_i\,,$$
$$i \in L \,:\, 0 = P_i - \sum\nolimits_{j=1}^{n} B_{ij} E_i E_j \sin(\theta_i - \theta_j)\,,$$
$$i \in L \,:\, 0 = Q_i + \sum\nolimits_{j=1}^{n} B_{ij} E_i E_j \cos(\theta_i - \theta_j)\,,$$

were $T_i > 0$, $K_i > 0$, and $F_i > 0$ are constants.

Exercise 7: Feasibility of the lossless power flow equations

Consider the power flow equations in polar form

$$P_i = \sum\nolimits_{j=1}^{n} B_{ij} E_i E_j \sin(\theta_i - \theta_j)\,,$$
$$Q_i = -\sum\nolimits_{j=1}^{n} B_{ij} E_i E_j \cos(\theta_i - \theta_j)\,,$$

where $i \in \{1, \ldots, n\}$. Assume that there are no shunt elements so that $B_{ii} = -\sum_{j=1, j\neq i}^{n} B_{ij}$. Show that the following conditions are necessary so that the lossless power flow equations have a solution:

a) active power: $\sum_{i=1}^{n} P_i = 0$; and

b) reactive power: $\sum_{i=1}^{n} Q_i \geq 0$.

Exercise 8: Feasibility of lossless power flow in the two-node case

Consider a lossless power grid consisting of a PV bus (source) connected to a PQ bus (load) through an inductive line, as illustrated in Figure 3. The power balance at the PQ node is given by

$$P = B\, E_{\text{source}}\, E_{\text{load}} \sin(\theta)\,,$$
$$Q = B\, E_{\text{load}}^2 - B\, E_{\text{source}}\, E_{\text{load}} \cos(\theta)\,,$$

where θ is the phase angle difference over the connecting line with susceptance B, E_{source} and E_{load} are the respective voltages, and P and Q are the active and reactive power demand, respectively.

Show that the power balance equations admit a solution if and only if the following condition holds

$$P^2 \,-\, B\, E_{\text{source}}^2 Q \,\leq\, B^2 E_{\text{source}}^4 \,/\, 4\,.$$

Think about the physical implications of this condition.

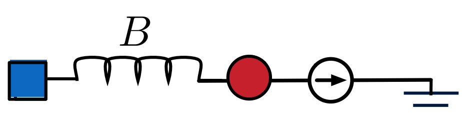

Figure 3: A lossless interconnection of a PV and a PQ bus

Exercise 9: Fallacies of decentralized integral control

Consider a power system model composed of n generators modeled by the swing equations

$$\dot{\theta} = \omega$$
$$M\dot{\omega} = -\, D\omega + P^* - P_{\text{e}} + u\,,$$

where $P^* \in \mathbb{R}^n$ accounts for the nominal power input by the prime movers as well as aggregated local loads, and $P_{\text{e}} \in \mathbb{R}^n$ is the vector of power injections into the grid with components

$$P_{\text{e},i}(\theta) = \sum_{j=1}^{n} B_{ij} \sin(\theta_i - \theta_j).$$

Finally, $u \in \mathbb{R}^n$ is a secondary control input to be desinged.

a) Assume that the above model has a synchronous solution $\dot{\theta}_i = \omega_{\text{sync}} \in \mathbb{R}$ for all $i \in \{1, \ldots, n\}$. Calculate ω_{sync} explicitly as a function of the problem parameters. Based on your result, elaborate on the task of secondary control?

b) Assume that a subset $\mathcal{K} = \{1, \ldots, k\} \subseteq \{1, \ldots, n\}$ of generators applies a decentralized integral control strategy

$$u_i(t) = -l_i \int_0^t \omega_i(\tau)\, d\tau\,, \quad i \in \mathcal{K}\,,$$

where $l_i > 0$, $i \in \mathcal{K}$, is a control gain. Characterize the set of equilibria.

c) Consider the same control strategy as in b), but now the frequency measurements are subject to a non-zero measurement bias $\eta_i \in \mathbb{R}$ for $i \in \mathcal{K}$:

$$u_i(t) = -l_i \int_0^t \omega_i(\tau) + \eta_i\, d\tau\,, \quad i \in \mathcal{K}\,,$$

Characterize the set of equilibria. Think about the cases where $|\mathcal{K}| = 1$ and $|\mathcal{K}| > 1$, that is, one or at least two decentralized integral controllers.

Exercise 10: Stability and power sharing in a droop-controlled microgrid

Consider an *acyclic* and connected microgrid consisting of inverter nodes I and loads L so that $I \cup L = \{1, \ldots, n\}$. The inverters perform droop control and the loads demand constant power

$$\begin{aligned} i \in I \,:\; & D_i \dot{\theta}_i = P_i^* - P_{\mathrm{e},i}(\theta) \\ i \in L \,:\; & 0 = P_i^* - \sum\nolimits_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)\,, \end{aligned}$$

where $P_{\mathrm{e},i}(\theta) = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$ is the measured power injection at inverter $i \in I$. Assume that $\sum_{i=1}^{n} P_i^* = 0$. Show the following:

a) Explain why the assumption $\sum_{i=1}^{n} P_i^* = 0$ is "without loss of generality".

b) Provide a parametric condition indicating whether the microgrid possesses a synchronous solution with angle differences satisfying $|\theta_i - \theta_j| < \pi/2$ for all branches $\{i, j\} \in \mathcal{E}$.

c) Analyze the local stability of the solution in b).

<u>*Hint:*</u> *If you want you can simplify problem c) by assuming that there are no load buses $L = \emptyset$ and all droop coefficients are unit-valued $D_i = 1$ for all $i \in I$.*

d) Assume that each inverter $i \in I$ has an injection constraint $P_{\mathrm{e},i}(\theta) \in [0, \overline{P_i}]$. Provide a condition on the control parameters P_i^* and D_i so that the inverters achieve fair proportional power sharing in a synchronous steady-state, that is, for all $i, j \in I$

$$\frac{P_{\mathrm{e},i}(\theta)}{\overline{P_i}} = \frac{P_{\mathrm{e},j}(\theta)}{\overline{P_j}}\,.$$

e) Assume that the total load $P_L = \sum_{i \in L} P_i^*$ can be served by the inverters with capacity constraints $\overline{P_i}$, $i \in I$, that is,

$$-\sum\nolimits_{i \in I} \overline{P_i} \leq P_L \leq 0\,.$$

Under this assumption and for the droop-coefficients chosen as in d), show that the inverters asymptotically meet their injection constraints $P_{\mathrm{e},i}(\theta) \in [0, \overline{P_i}]$ for all $i \in I$.

<u>*Hint:*</u> *the exercises b)-d) can be solved independently from another.*

Exercise 11: $\mathcal{H}_2$ performance of consensus systems

The following exercise is a primer for calculating $\mathcal{H}_2$ norms of linearized power system models. Consider the continuous-time consensus dynamics with disturbance

$$\dot{x}(t) = -Lx(t) + w(t),$$

where $L = L^\top$ is the Laplacian matrix of an undirected and connected graph and $w(t)$ is an exogenous disturbance input signal. You can think about these dynamics as a set of linearized swing equations in the limits of zero mass and unit damping.

Pick a matrix $Q \in \mathbb{R}^{p\times n}$ satisfying $Q\mathbb{1}_n = \mathbb{0}_p$ and define the output signal

$$y(t) = Qx(t) \in \mathbb{R}^p$$

as the solution from zero initial conditions $x(0) = \mathbb{0}_n$. Define the *system* $\mathcal{H}_2$ *norm* from w to y by

$$\|\mathcal{H}\|_2^2 = \int_0^\infty y(t)^\top y(t)dt = \int_0^\infty x(t)^\top Q^\top Qx(t)dt = \text{trace}\left(\int_0^\infty H(t)^\top H(t)dt\right),$$

where $H(t) = Qe^{-Lt}$ is the so-called *impulse response matrix*.

a) Show $\|\mathcal{H}\|_2 = \sqrt{\text{trace}(P)}$, where P is the solution to the Lyapunov equation

$$LP + PL = Q^\top Q.$$

b) Show $\|\mathcal{H}\|_2 = \sqrt{\text{trace}\left(L^\dagger Q^\top Q\right)/2}$, where $L^\dagger$ is the pseudoinverse of L.

c) Define *short-range* and *long-range output matrices* Q_{sr} and Q_{lr} by $Q_{\text{sr}}^\top Q_{\text{sr}} = L$ and $Q_{\text{lr}}^\top Q_{\text{lr}} = I_n - \frac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top$, respectively. Show:

$$\|\mathcal{H}\|_2^2 = \begin{cases} n-1, & \text{for } Q = Q_{\text{sr}}, \\ \sum_{i=2}^n \frac{1}{\lambda_i(L)}, & \text{for } Q = Q_{\text{lr}}. \end{cases}$$