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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4	5	Exercise
8	5	4	4	4	25 Points

Consider the following circuit:

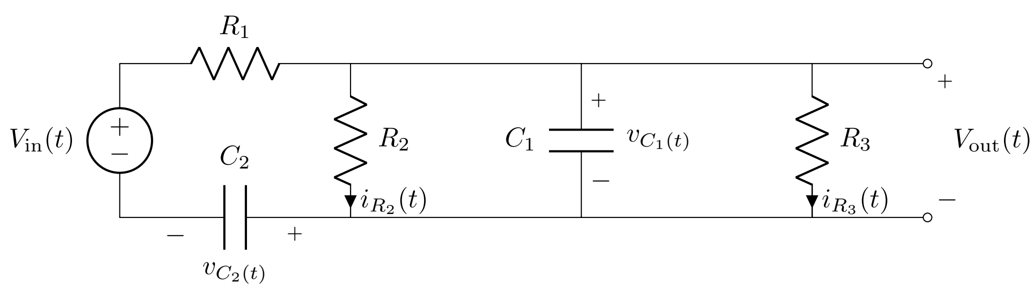

Figure 1: Electrical circuit

1. Using $x(t) = \begin{bmatrix} v_{C_1}(t) & v_{C_2}(t) \end{bmatrix}^T$ as a state vector, $u(t) = V_{\text{in}}(t)$ as an input and $y(t) = V_{\text{out}}(t)$ as an output, describe the system in state-space form by finding the matrices A, B, C, D in
$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{1}$$

2. Assume now that $R_1 = 1$, $R_2 = R_3 = 2$ and $C_1 = C_2 = 1$. Argue from physical intuition that the system is asymptotically stable. Verify this by analyzing the state equations derived in Part 1.

3. For the system parameters given in Part 2 propose a matrix Q for a quadratic Lyapunov function
$$V(x(t)) = \frac{1}{2} x(t)^T Q x(t)$$

that can be used to verify that the system is asymptotically stable. Compute the matrix R that satisfies the Lyapunov equation

$$A^T Q + QA = -R \tag{2}$$

and confirm that Q and R satisfy the conditions that ensure asymptotic stability.

4. By setting $f(x(t)) = Ax(t)$ verify that your Lyapunov function from Part 3 also satisfies the conditions of the more general Lyapunov theorem for nonlinear systems.

5. You build the circuit, but discover that due to a manufacturing defect resistor R_2 has been short-circuited ($R_2 = 0$). Is the circuit still asymptotically stable? Is it observable? You can argue from physical intuition without re-deriving the state equations.

Exercise 2

1	2	3	4	5	Exercise
6	4	4	8	3	25 Points

Consider the following system with $x(t) \in \mathbb{R}^3$, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}^2$:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} \frac{1}{100} & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{B} u(t) \tag{3}$$

1. Write down the controllability matrix and determine whether the system is controllable.

2. Would controllability change if the second entry of B was 0? Argue either mathematically or using the dependencies between the states.

3. Is the system stable? Is it asymptotically stable?

4. Your friend from EPFL argues that three states is too much and wants to remove one of the states. He proposes to either remove x_1 since "it only has a coefficient of 0.01", or to remove x_2 since "it does not depend on other states". He hence comes up with two new models with two states each:

$$\begin{aligned} \dot{\tilde{x}}(t) &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tilde{u}(t) \\ \tilde{y}(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \tilde{x}(t), \end{aligned} \tag{4}$$

and

$$\begin{aligned} \dot{\bar{x}}(t) &= \begin{bmatrix} \frac{1}{100} & 0 \\ 0 & -1 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \bar{u}(t) \\ \bar{y}(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{x}(t). \end{aligned} \tag{5}$$

For each of the two systems (4) and (5), argue why they are a poor approximation of the original system (3).

5. If your friend would have picked the more standard approach of changing the state using an invertible transform $\tilde{x}(t) = Tx(t)$ with $T \in \mathbb{R}^{3\times 3}$ and invertible (i.e. without removing any states), could any of the problems you detected in part 4 have occurred? Justify your answers.

Exercise 3

1	2	3	4	5	Exercise
3	4	5	5	8	25 Points

Consider the continuous-time system

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + x_1(t)x_2(t) \\ \dot{x}_2(t) &= -x_2(t) \end{aligned} \tag{6}$$

where $x_1(t) \in \mathbb{R}$ and $x_2(t) \in \mathbb{R}$.

1. Is the system linear? Is it autonomous? Is it time-invariant?

2. Show that the system has a unique equilibrium at the origin. Determine its stability using linearization.

3. Consider the function
$$V_1(x_1, x_2) = x_1^2 + x_2^2.$$
Show that there exists $\epsilon > 0$ such that the Lie derivative of V_1 along (6) is negative for all points in the set
$$\mathcal{S} = \{(x_1, x_2) \in \mathbb{R}^2 | \; x_1^2 + x_2^2 < \epsilon, (x_1, x_2) \neq (0, 0)\}.$$
What can you conclude about the stability of (6) from this? What can you conclude about the domain of attraction of the equilibrium?

4. For the function V_1, show that there exist points $(x_1, x_2) \in \mathbb{R}^2$ where the Lie derivative of V_1 along (6) is positive. Does this mean that the origin is not globally asymptotically stable for system (6)?

5. Now consider the function
$$V_2(x_1, x_2) = \ln(1 + x_1^2) + x_2^2,$$
where $\ln(\cdot)$ represents the natural logarithm.
   (a) Show that $V_2(x_1, x_2) \geq 0$ for all (x_1, x_2) and it is zero only at the equilibrium point of (6).
   (b) Show that the Lie derivative of V_2 along (6) is negative for all $(x_1, x_2) \in \mathbb{R}^2$ except at the equilibrium of (6). *Hint: Use the identity* $(x_1 - x_1x_2)^2 = x_1^2 - 2x_1^2x_2 + x_1^2x_2^2$.
   (c) What can you conclude about the asymptotic stability of the origin?

Exercise 4

1	2	3	4	Exercise
4	7	7	7	25 Points

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

given by the matrices

$$A = \begin{bmatrix} -d & -k \\ 1 & 0 \end{bmatrix} \ , \ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ , \ C = \begin{bmatrix} 0 & 1 \end{bmatrix} \ , \ D = 0.$$

and real numbers $d \in \mathbb{R}$ and $k \in \mathbb{R}$.

1. Compute the transfer function $G(s)$ from $u(t)$ to $y(t)$. For which values of k and d is the transfer function $G(s)$ stable?

2. For the parameter range for which the system is stable, assume that $k > \frac{1}{2}d^2$ and compute the frequency $\omega^\star$ at which the maximum of the magnitude $\|G(j\omega)\|$ occurs.

3. Figure 2 shows the zero-state response for four linear time-invariant systems and the input signal

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ \sin(2t) & \text{if } t \geq 0. \end{cases}$$

   For each of the three transfer functions

   (i) $G_1(s) = \frac{-6s}{s^2+6s+4}$

   (ii) $G_2(s) = \frac{2(s^2+4)}{(s+1)^2}$

   (iii) $G_3(s) = \frac{1}{\frac{1}{4}s^2+\frac{1}{4}s+1}$

   the zero-state response is shown in Figure 2, one plot shows the zero-state response of a transfer function not given above. Which transfer function corresponds to which zero-state response? Justify your answer. *Hint: Use the magnitudes* $\|G_i(j\omega)\|$ *for* $i = \{1, 2, 3\}$ *at a specific frequency.*

4. Consider the system Σ_1 given by the transfer function

$$G_{\Sigma_1}(s) = \frac{1}{(s+2)(s-1)(s+10)}$$

   connected to a controller $\Sigma_2 : y_2(t) = Ke(t)$ as shown in Figure 3. Moreover, Figure 4 shows the Nyquist diagram of the open-loop system Σ_1. For which values of $K \in \mathbb{R}$ is the closed-loop system shown in Figure 3 stable? Assume that the curve intersects itself at the point $p = -\frac{1}{108}$.

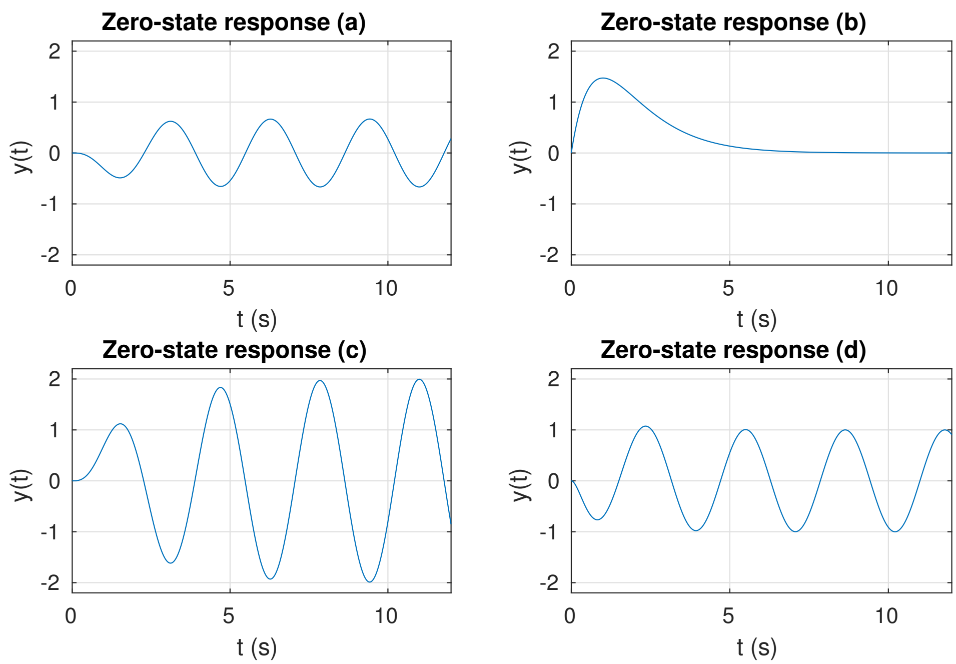

Figure 2: Zero-state-responses for input signal $u(t)$.

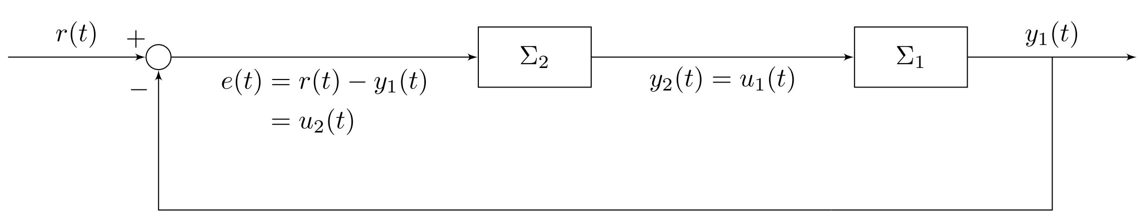

Figure 3: Feedback system

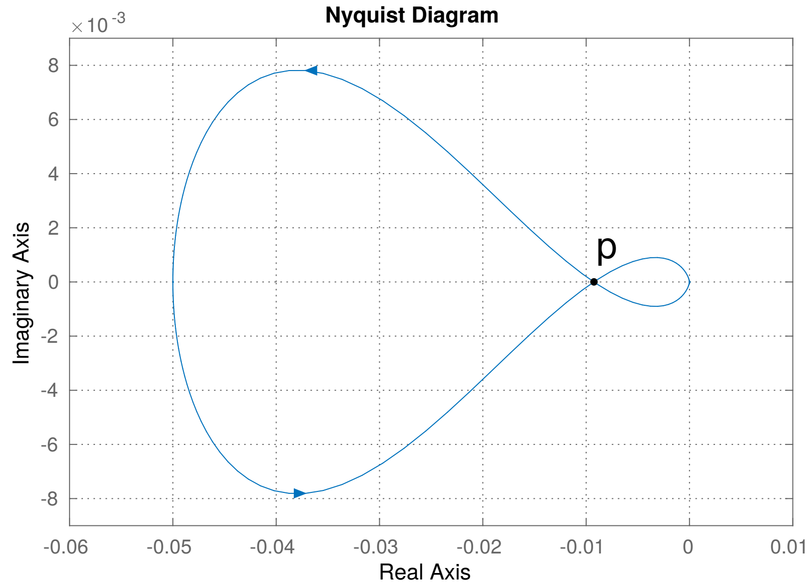

Figure 4: Nyquist diagram of the open-loop transfer function of Σ_1.