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D-ITET
Winter 2011/12
23.01.2012

Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4	Exercise
4	5	8	8	25 Points

Consider the following system

$$\dot{x}_1(t) = -x_2(t)e^{x_1(t)x_2(t)}$$
$$\dot{x}_2(t) = x_1(t) + ke^{x_2(t)},$$

where $k \in \mathbb{R}$ is a constant.

1. What is the dimension of the system? Is the system linear? Is the system autonomous? Is the system time invariant?
2. Determine the equilibrium points of the system.
3. Using linearization, determine the stability of the equilibrium points whenever possible. How does stability depend on the value of k.
4. For the case $k = 0$, examine the stability of the equilibrium using the Lyapunov function $V(x_1(t), x_2(t)) = x_1(t)^2 + x_2(t)^2$.
(Hint: $\alpha(1 - e^{\alpha}) \leq 0$ for all $\alpha \in \mathbb{R}$).

Exercise 2

1	2	3	4	Exercise
5	6	8	6	25 Points

Consider the discrete time system

$$\begin{aligned}\theta(k+2) + 1.5\theta(k+1) + 3\theta(k) - \theta(k)^2 &= u(k) + v(k)\\ y(k) &= \theta(k+1)\end{aligned}$$

where $u(k)$ and $v(k)$ are input signals and $y(k)$ is an output signal.

1. Using $x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} \theta(k) \\ \theta(k+1) \end{bmatrix} \in \mathbb{R}^2$ as a state vector, write the system in state space form

$$\begin{aligned}x(k+1) &= f(x(k), u(k), v(k))\\ y(k) &= h(x(k), u(k), v(k)).\end{aligned}$$

   What is the dimension of the system? Is the system linear?

2. For $u(k) = 0$ and $v(k) = 0$, determine all equilibria of the system.

3. For $u(k) = 0$, let $v(k) = a + bx_1(k) + cx_1(k)^2$. Select values for the constants a, b, and c so that the resulting autonomous system is linear, has a unique equilibrium at the origin, and has eigenvalues -0.5 and -1. Is the resulting system stable? Is it asymptotically stable?

4. Consider the original system with the input $v(k)$ as determined in part 3. Is the system observable? Is the system controllable from the input $u(k)$?

Exercise 3

1	2	3	Exercise
9	7	9	25 Points

Consider the autonomous LTI system

$$\dot{x}(t) = Ax(t)$$

1. Let A be

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

   Show that the corresponding state transition matrix $\Phi(t)$ is

$$\Phi(t) = e^{-t} \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \quad .$$

   Compute the eigenvalues of the matrix A. Is the system stable? Is the system asymptotically stable?

2. Now, let A be

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Compute the corresponding state transition matrix $\Phi(t)$. Compute the eigenvalues of the matrix A. Is the system stable? Is the system asymptotically stable?

3. Finally, let A be

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Compute the corresponding state transition matrix $\Phi(t)$. Compute the eigenvalues of the matrix A. Is the system stable? Is the system asymptotically stable?
   (Hint: Note that $A^3 = 0$ in this case.)

Exercise 4

1	2	3	4	5	Exercise
7	4	4	4	6	25 Points

Consider the following system:

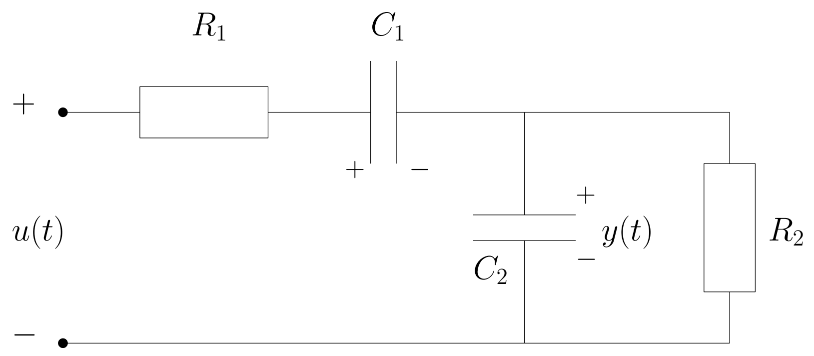

Figure 1: Electrical circuit

1. Using $u(t)$ as an input and $y(t)$ as an output. Define an appropriate state vector $x(t)$ and derive the state space equations of the system. What is the dimension of the system?

2. Assume $C_1 = C_2 = 1F$ and $R_1 = R_2 = 1\Omega$. Is the system controllable? Is the system observable? Is the system stable? Justify your answers mathematically.

3. You buy such a circuit board with $C_1 = C_2 = 1F$ and $R_1 = 1\Omega$, which can be connected to an external load resistor R_2. In the users manual the manufacturer warns not to use very high loads R_2 to avoid controllability problems. Provide the mathematical justification for this warning.

4. Calculate the transfer function $G(s)$ for the values given in part 2.

5. For the system of part 2, you want to use a state feedback controller $u(t) = Kx(t)$ to place the eigenvalues of the system both at -1. Compute the gain matrix $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ that may achieve this.