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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	Aufgabe
8	6	11	25 Punkte

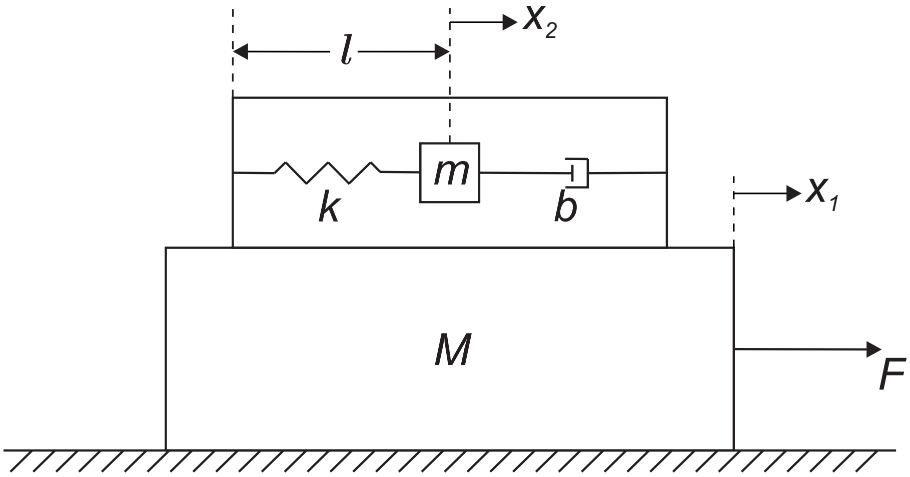

Figure 1: A mechanical accelerometer mounted on a mass M.

A mechanical accelerometer is used to measure the acceleration of a moving object as shown in Figure 1. The accelerometer consists of a mass m which is attached to a case by means of a spring of stiffness k and a damper of constant b. The accelerometer uses the relative position x_2 of mass m, with respect to the accelerometer case, to estimate the acceleration of the mass M. Let x_1 denote the horizontal position of mass M and x_2 the relative position of mass m measured from the equilibrium length of the spring when an external force F is applied on the mass M. Assume that the contact between the mass M and the supporting surface is frictionless, the equilibrium length of the spring is l and that mass m is much smaller than mass M ($m << M$), so that the force applied from m to M through the spring and damper can be neglected.

1. Derive a state space model for this system using $u = F$ as the input and $y = x_2$ as the output.

2. Derive the transfer function $G(s)$ from the input u to the output y. (Hint: You do not have to invert the matrix $(sI - A)$, you can use one of the state equations instead.)

3. A step input of magnitude $F = 10\ Nt$ is applied to the system at time $t = 0$. Using the values $M = 5\ Kg$, $m = 1\ Kg$, $k = 4\ Nt/m$ and $b = 4\ Kg/sec$ calculate the expression of $y(t)$ as a function of time for zero initial condition ($x_1(0) = \dot{x}_1(0) = 0$ and $x_2(0) = \dot{x}_2(0) = 0$).

Exercise 2

1	2	3	4	Exercise
5	5	5	10	25 Points

Consider the continuous time, linear, time invariant system:

$$\dot{x} = \begin{bmatrix} 0 & a+1 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (1)$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} + u \quad (2)$$

where the variable a is a real valued constant.

1. For which values of a is the system controllable?
2. For which values of a is the system observable?
3. Assume that $u(t) = 0$ is applied to the system for all time. For which values of a is the resulting system stable? For which values is it asymptotically stable? For which values is it unstable?
4. Consider now a state feedback controller $u = Kx = \begin{bmatrix} k_1 & k_2 \end{bmatrix} x$. Design a gain matrix K such that the poles of the closed loop system are both at -1 whenever possible. Is this possible for all values of a? Why?

Exercise 3

1	2	3	4	Aufgabe
5	5	10	5	25 Punkte

Consider the following differential equation, where $\alpha \in \mathbb{R}$ is a constant:

$$\ddot{\theta} = \sin(-\theta + \alpha\dot{\theta}) \tag{3}$$

1. Derive the state-space representation of the system.
2. Determine all the equilibrium points of the system.
3. Using linearization, determine the values of α for which the equilibrium points found in Part 2 are asymptotically stable, or unstable. Are there cases where the linearization is inconclusive?
4. For the case $\alpha = 0$, deduce that the equilibrium $\theta = 0, \dot{\theta} = 0$ is stable by using an appropriate Lyapunov function.

Exercise 4

1	2	3	Aufgabe
5	10	10	25 Punkte

You are given the discrete time, linear, time invariant system:

$$x_{k+1} = Ax_k \qquad x_k \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}$$

1. If matrix A is diagonalizable, show that for k=1,2,...

   $$A^k = S\Lambda^k S^{-1}$$

   where

   $$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

   and $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of the matrix A.

2. Show that if the matrix S has columns, $s_1, s_2, ..., s_n$ and the matrix S^{-1} has rows $e_1^T, e_2^T, ..., e_n^T$, then

   $$A^k = \sum_{i=1}^{n} \lambda_i^k s_i e_i^T \tag{4}$$

3. Discuss in detail the stability of the system as a function of λ_i using your answer in Part 2. Provide proofs to support your answers.