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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	**2**	**3**	**4**	**5**	**Exercise**
6	**6**	**1**	**6**	**6**	**25 Points**

A loudspeaker and its driving circuitry are shown schematically in Figure 1. The horizontal displacement of the loudspeaker cone with equivalent mass of m is denoted by $z(t)$. R denotes the resistance of the loudspeaker coil, while L the coil's inductance.

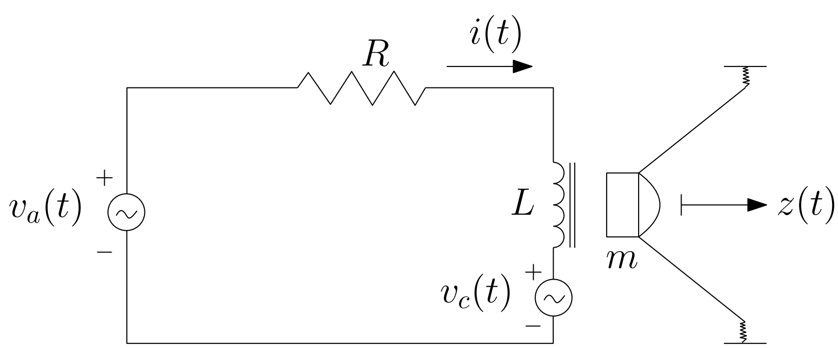

Figure 1: Schematic of a simplified electromechanical model of a loudspeaker.

The loudspeaker cone is driven by the force

$$F(t) = \ell\, i(t), \tag{1}$$

where $i(t)$ is the current through the coil and ℓ a coil parameter. The coil voltage $v_c(t)$ is related to the motion of the loudspeaker cone and is given by

$$v_c(t) = \ell\, \dot{z}(t). \tag{2}$$

The air resistance to the cone movement is proportional to the cone velocity with coefficient d. Assume that all parameters are positive $m, R, L, \ell, d > 0$ and that the vertical displacement of the cone and all other forces acting on it are negligible.

1. Choosing as state $x(t) = \begin{bmatrix} z(t) & \dot{z}(t) & i(t) \end{bmatrix}^T$, the applied voltage $v_a(t)$ as input and the cone displacement $z(t)$ as output show that the state-space model for the system can be described by the following system matrices.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{d}{m} & \frac{\ell}{m} \\ 0 & -\frac{\ell}{L} & \frac{R}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}.$$

2. Is the system observable? Is it controllable?

3. Calculate the equilibrium of the system under a constant zero input voltage $v_a(t) = 0$.

4. State the conditions on the parameter values under which the system is stable and asymptotically stable.

5. Consider now a state feedback controller of the form

$$u(t) = K\,x(t) = \begin{bmatrix} k_1 & \frac{\ell}{L} & k_3 \end{bmatrix} x(t). \tag{3}$$

Derive conditions on k_1 and k_3 such that the closed-loop system $\dot{x} = (A + BK)\,x(t)$ is asymptotically stable.

Hint: The third-order polynomial $P(s) = s^3 + a_2 s^2 + a_1 s + a_0$ has roots with negative real part if and only if its coefficients are positive ($a_0, a_1, a_2 > 0$) and $a_2 a_1 > a_0$.

Exercise 2

1	2	3	4	5	Exercise
4	4	7	5	5	25 Points

Consider the following linear time invariant system with parameters $a, b, c \in \mathbb{R}$:

$$\dot{x}(t) = \overbrace{\begin{bmatrix} -1 & 4 \\ a & -2 \end{bmatrix}}^{A} x(t) + \overbrace{\begin{bmatrix} 0 \\ b \end{bmatrix}}^{B} u(t)$$
$$y(t) = \underbrace{\begin{bmatrix} c & 1 \end{bmatrix}}_{C} x(t).$$

1. For what values of (a, b, c) does the equation $A^T P + PA = -I$ have a unique symmetric positive definite solution P? What does this imply for the zero-input-response of the system?

2. For what values of (a, b, c) is the system controllable?

For the remainder of the exercise, assume $(a, b, c) = (3, 1, 1)$.

3. Consider a feedback controller of the form

$$u(t) = Kx(t) = \begin{bmatrix} 0 & k \end{bmatrix} x(t).$$

   Assume you want the closed-loop eigenvalues, λ_1 and λ_2, of $\dot{x}(t) = (A + BK)x(t)$ to satisfy $\lambda_2 = \alpha\lambda_1$, for $\lambda_1 < 0$ and $\alpha > 0$. For which values of $\alpha > 0$ can we always find a $k \in \mathbb{R}$ so that this condition is satisfied?
   Hint: Write down the characteristic polynomial of $(A + BK)$ and compare it with a polynomial whose roots are λ_1 and $\alpha\lambda_1$. Remember that k is a real number and a quadratic with real coefficients has real roots if and only if its discriminant is non-negative. You may assume that $0.5\sqrt{2496} \approx 24.98$.

4. Take now $\alpha = 0.02$ in Part 3. Find the k that leads to the fastest response of the closed loop system $\dot{x}(t) = (A + BK)x(t)$.

5. You would like to implement the controller from Part 4 when you are only able to measure the output $y(t)$ instead of the whole state $x(t)$. Is it possible to design an observer to infer the states from the output? You may answer without deriving the equations for the observer.

Exercise 3

1(a)	1(b)	2	Exercise
5	10	10	25 Points

1. Consider the system whose block diagram is shown in Figure 2. Assume $H(s) = \frac{1}{s-2}$, $F(s) = \frac{as+5}{s+2}$, and $a \in \mathbb{R}$.

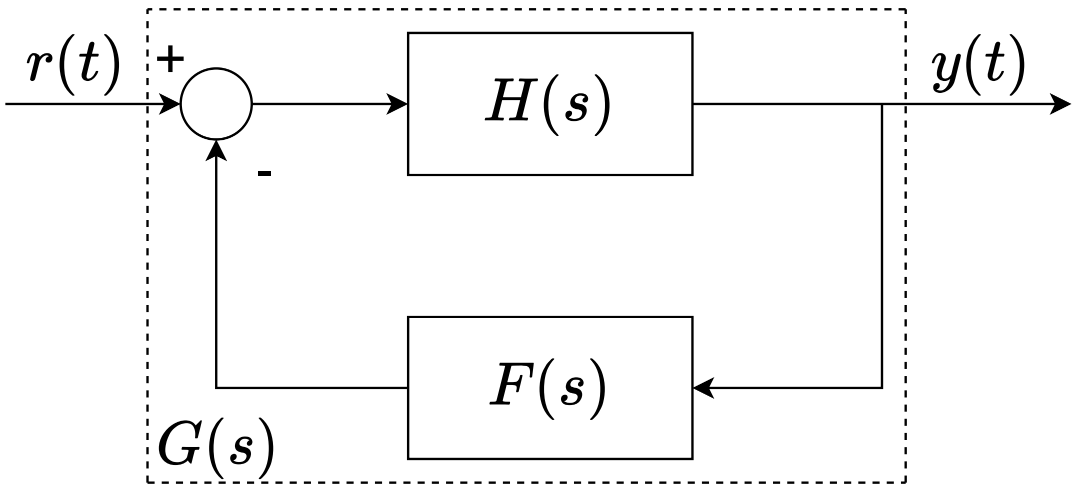

Figure 2: Block diagram.

(a) Verify that the transfer function $G(s)$ from the input $r(t)$ to the output $y(t)$ is given by

$$G(s) = \frac{s+2}{s^2 + as + 1}$$

(b) Consider $a \in \{-2, 0, 0.5, 2\}$. Match the step responses of $G(s)$ in Figure 3 with the corresponding value of the parameter a. Justify your answer.

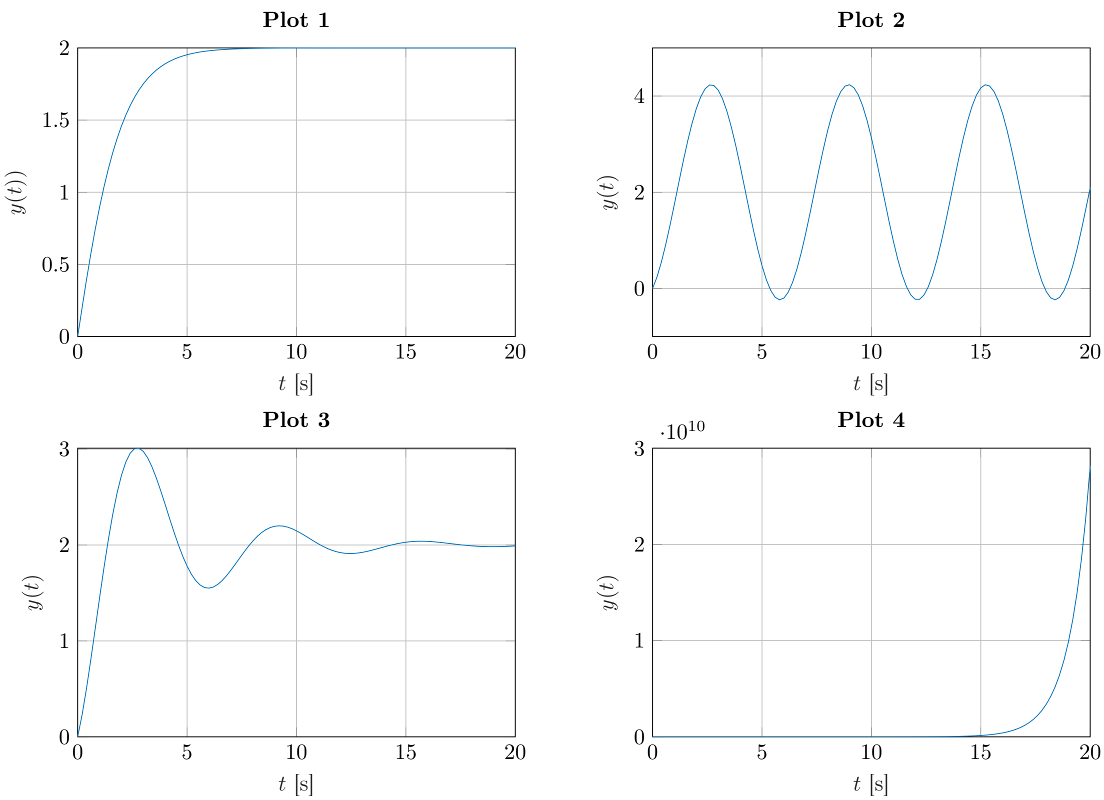

Figure 3: Step responses of the system $G(s)$ for different values of the parameter a.

2. The system $G(s)$ in Part 1 with $a = -2$ is to be controlled via a proportional gain $K \in \mathbb{R}$ as in Figure 4. Given the Nyquist plot in Figure 5, for which values of K will the closed loop system be stable?

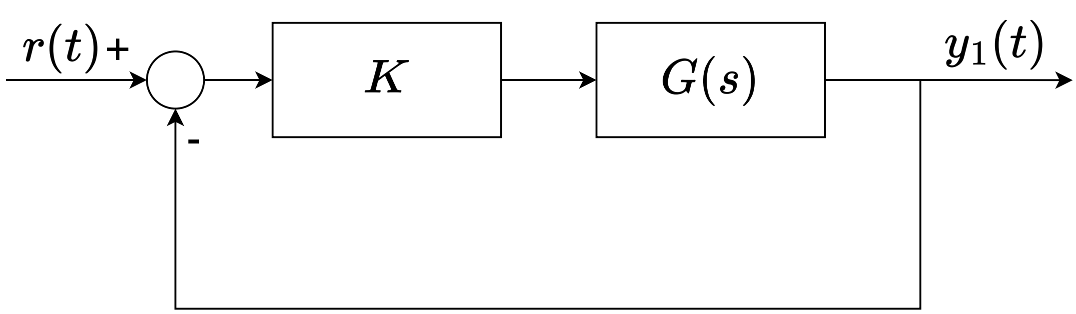

Figure 4: Block diagram.

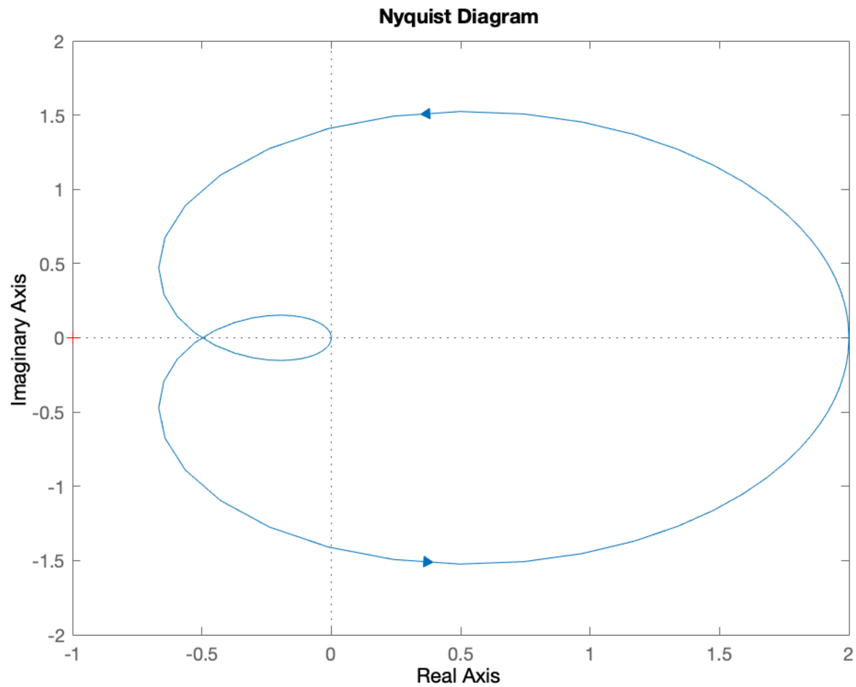

Figure 5: Nyquist plot of the system $G(s)$ for $a = -2$.

Exercise 4

1	2	3	4	5	Exercise
6	1	5	8	5	25 Points

Consider the system given by the following state-space equations

$$\dot{x}_1(t) = x_1(t)x_2(t),$$
$$\dot{x}_2(t) = -kx_2(t) - x_1^2(t) + a$$

with state variables $x_1(t), x_2(t) \in \mathbb{R}$ and constants $a, k \in \mathbb{R}$.

1. Derive all equilibrium points of the system. How many equilibrium points does the system have for $a > 0$ and $a < 0$?

2. Verify that when $a = 0$, $\hat{x} = (0, 0)$ is the unique equilibrium of the system.

3. For $a = 0$ what can you say about the stability of the equilibrium $\hat{x} = (0, 0)$ for different values of $k \in \mathbb{R}$ using linearisation?

4. Can you use the Lyapunov function $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ to show stability of the equilibrium $\hat{x} = (0, 0)$ for $a = 0$ and $k > 0$? If yes, prove stability, if not justify your answer! Can you use the same Lyapunov function to prove local asymptotic stability for the same values of a and k?

   Hint: You may take $S = \mathbb{R}^2$ as the open set used in the theorem.

5. For $K \geq 0$ consider the set $S_K = \{x(t) \in \mathbb{R}^2 \,|V(x) \leq K\}$. Using the Lyapunov function $V(x)$ from Part 4, argue that the set S_K is invariant when $a = 0$ and $k > 0$ for any $K \geq 0$. Hence, using LaSalle's theorem, show that the equilibrium $\hat{x} = (0, 0)$ is globally asymptotically stable.

   Hint: You may assume that the set S_K is compact for all $K \geq 0$.