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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3(a)	3(b)	3(c)	4	Exercise
3	4	3	3	6	6	25 Points

Your task is to design a lane keeping state feedback controller for the bicycle model shown below.

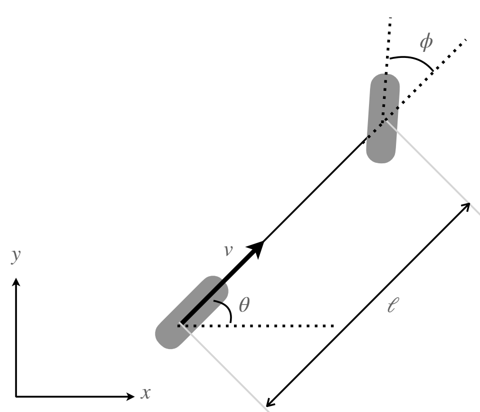

Figure 1: Bicycle model with corresponding model parameters. (x, y) position of rear wheel axle in the global coordinate frame, v: speed of rear wheel in the direction it is pointing (forward velocity), ϕ: steering angle, ℓ: bicycle length, θ: angle of the bicycle with respect to the x axis.

The rear axle kinematics model of the bicycle in Figure 1 is given by

$$\dot{x} = v\cos(\theta) \tag{1}$$
$$\dot{y} = v\sin(\theta) \tag{2}$$
$$\dot{\theta} = \frac{v}{\ell}\tan(\phi) \tag{3}$$

Throughout the exercise, we assume that we are operating at constant forward velocity $v = \bar{v}$ and are interested in controlling the steering angle, θ, so that the bicycle tracks the lane.

1. Linearize the dynamics (1)-(3) around $\theta = 0$, using $u = \tan(\phi)$ as the input. Write down the resulting dynamics.

   Hint: $\cos(\alpha) \simeq 1,\ \sin(\alpha) \simeq \alpha$ for small α.

2. Use the forward Euler approximation with the discretization time δ to derive the discrete-time dynamics in the state $z_k = [x_k\ y_k\ \theta_k]^T$ and input u_k. Write out the resulting system of equations.

3. For lane keeping, we would like the bicycle to move forward in the x direction with a constant offset $\bar{y}$ in the y direction. This means that we would like to track a reference trajectory $z_k^r = [x_k^r\ y_k^r\ \theta_k^r]^T$ given by

$$x_{k+1}^r = x_k^r + \delta\bar{v}, \tag{4}$$
$$y_k^r = \bar{y}. \tag{5}$$
$$\theta_k^r = 0, \tag{6}$$

   For the rest of the exercise, set $\delta = 0.2$, $\bar{v} = 10$, and $\ell = 1$.

   (a) Using the model in Part 2., show that the dynamics of the tracking error $\tilde{z}_k = z_k - z_k^r = \left[\tilde{x}_k\ \tilde{y}_k\ \tilde{\theta}_k\right]^T$ are given by

$$\begin{bmatrix} \tilde{x}_{k+1} \\ \tilde{y}_{k+1} \\ \tilde{\theta}_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_k \\ \tilde{y}_k \\ \tilde{\theta}_k \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u_k. \tag{7}$$

   (b) Is the system with zero input asymptotically stable?

   Hint: Remember that this is a discrete time system!

   (c) Show that the resulting system is uncontrollable. Determine the reachable space and the uncontrollable modes. Is the system stabilisable? Which of the modelling approximations introduced above could have caused this loss of controllability?

   Hint: Eigenvalues with negative real part in continuous time correspond to eigenvalues with magnitude less than one in discrete time.

4. Finally, we want to design a state feedback controller of the form

$$u_k = \begin{bmatrix} 0 & \frac{p}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \tilde{x}_k \\ \tilde{y}_k \\ \tilde{\theta}_k \end{bmatrix}. \tag{8}$$

   To stabilise the states $(\tilde{y}_k, \tilde{\theta}_k)$ in the error dynamics (8). p is a controller gain, to be determined. Determine the value of p so the closed-loop system has two poles at 0.5 and one pole at 1. Comment on the stability of the resulting discrete-time system.

Exercise 2

1(a)	1(b)	1(c)	1(d)	2(a)	2(b)	2(c)	Exercise
3	**3**	**4**	**3**	**4**	**4**	**4**	**25 Points**

1. Consider the following dynamical system:

$$\begin{aligned}\dot{x}_1(t) &= ax_1(t) + bx_2(t) + u(t)\\ \dot{x}_2(t) &= cx_1(t) + dx_2(t)\\ y(t) &= x_1(t).\end{aligned}$$

   (a) For which values of a, b, c, d is the system observable?

   (b) For which values of a, b, c, d is the system controllable?

   (c) Consider the case $a = 1$, $b = 0$, $c = -2$, $d = -1$ and a feedback law $u(t) = Kx(t)$ with $K = \begin{bmatrix} k & 0 \end{bmatrix}$. For which values of k is the closed-loop system asymptotically stable? Which values of k lead to the fastest convergence rate?

   (d) Thinking back to your answer in Part (a), can the controller that achieves the fastest convergence rate in Part (c) be implemented using output feedback and an observer?

2. Consider the state-space system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$ defined by:

$$A = \begin{bmatrix} 2 & 6 & 0 \\ 0 & -8 & 0 \\ 6 & -4 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix}^\top \qquad C = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \qquad D = 0.$$

   (a) Is the system stable?
   Hint: Start the determinant calculation with the last column of the matrix.

   (b) Is the system detectable?

   (c) Is the system stabilisable?

Exercise 3

1(a)	1(b)	1(c)	1(d)	2(a)	2(b)	Exercise
4	4	4	8	3	2	25 Points

1. Consider the dynamical system defined with the following differential equation

$$\ddot{z}(t) + a\dot{z}(t) + bz(t) = u(t), \tag{9}$$

where $u(t)$ is the control input and the measured output of the system is $y(t) = \dot{z}(t)$.

(a) By defining an appropriate state $x(t)$, write the system in the state-space form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

(b) Show that the transfer function of the system is

$$G(s) = \frac{s}{s^2 + as + b}.$$

(c) Use $G(s)$ to compute the impulse response of the system for $(a, b) = (4, 0)$ and the step response for $(a, b) = (0, 4)$?

Hint: Recall that the step and impulse responses are *zero state* responses of the system to a step and an impulse input respectively.

(d) Figure 2 shows the Bode plots of four transfer functions $G_1(s)$, $G_2(s)$, $G_3(s)$ and $G_4(s)$. Determine which transfer function corresponds to which of the following values of a and b:

i. $(a, b) = (2, -3)$,
ii. $(a, b) = (3, 0)$,
iii. $(a, b) = (4, 3)$,
iv. $(a, b) = (0.1, 9)$.

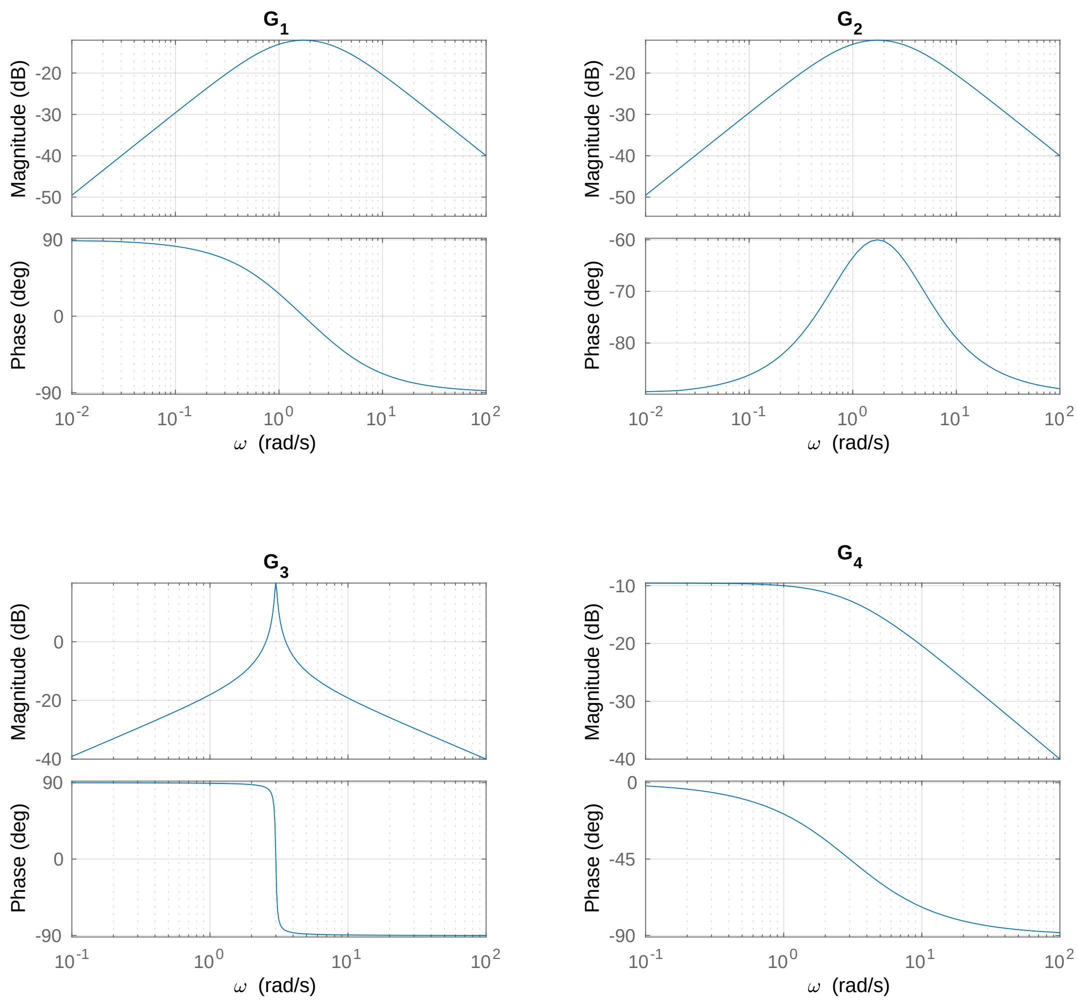

Figure 2: Bode plots of the transfer function $G(s)$ for different values of the parameters a and b.

2. The transfer function $G_l(s) = \frac{s+1}{s^2+3s+9}$ is put in a feedback loop with a proportional controller with gain K. The Nyquist plot of $G_l(s)$ is given in Figure 3.

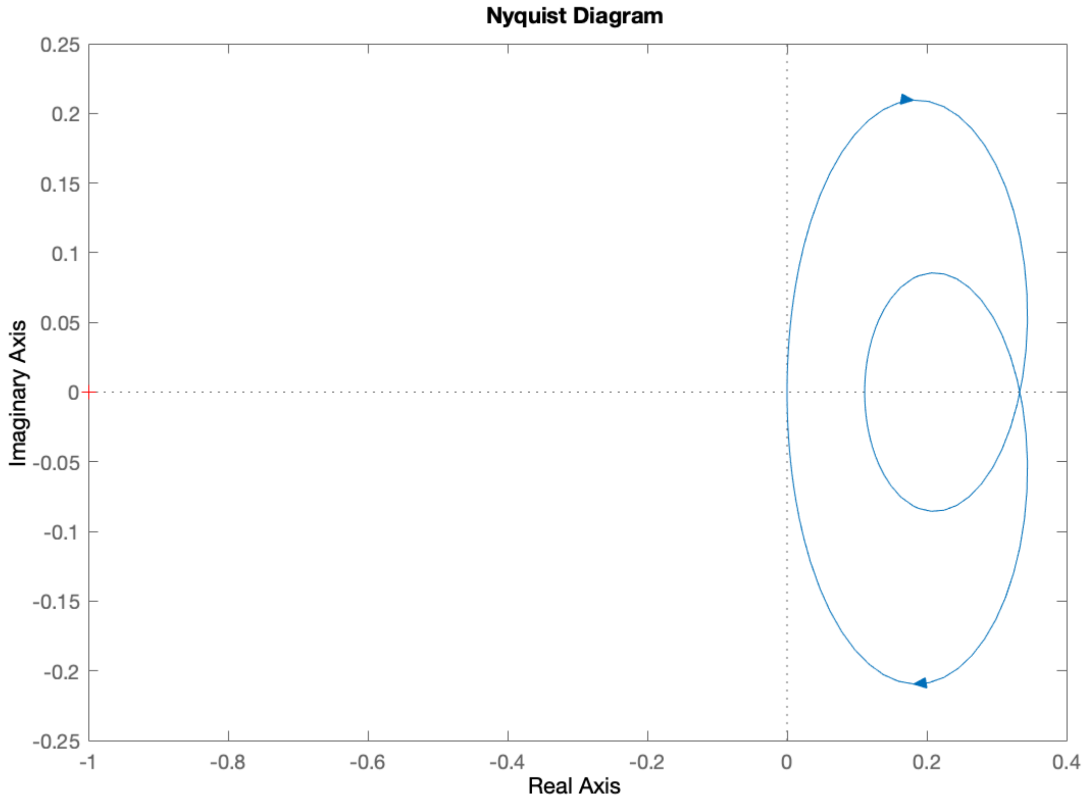

Figure 3: Nyquist plot of $G_l(s)$.

(a) Is the closed loop system stable for the control gain $K = -2$?

(b) How will the decrease of the gain K affect the stability of the system?

Exercise 4

1	2	3	4(a)	4(b)	4(c)	4(d)	Exercise
3	4	4	4	2	4	4	25 Points

Consider the following system with parameters $a > 0$, and $b > 0$

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + bx_1(t)x_2(t)\\ \dot{x}_2(t) &= -2x_2(t) - bx_1(t)x_2(t) + a.\end{aligned}$$

1. Is the system linear? Is it time-invariant? Is the system autonomous? Provide a short justification for each of these points.

2. Show that if $a \neq \frac{2}{b}$ the system has two equilibria at $\overline{x} = \begin{bmatrix} 0 \\ \frac{a}{2} \end{bmatrix}$ and $\tilde{x} = \begin{bmatrix} a - \frac{2}{b} \\ \frac{1}{b} \end{bmatrix}$.

3. For $a \neq \frac{2}{b}$ analyze the stability of the two equilibria using linearization.

   Hint: Stability may depend on the relation between a and b.

4. Note that if $a = \frac{2}{b}$ there is a unique equilibrium at $\overline{x} = \begin{bmatrix} 0 \\ \frac{a}{2} \end{bmatrix}$. Consider this case and assume that initially $x_1(0) \geq 0$ and $x_2(0) \geq 0$.

   (a) Show that the set $S = \{x \mid x_1 \geq 0, x_2 \geq 0\}$ is invariant.

   Hint: Consider what happens on the boundary of the set, where either $x_1 = 0$ or $x_2 = 0$.

   (b) Can you determine the stability of the equilibrium $\overline{x}$ using linearization?

   (c) Consider the Lyapunov function

   $$V(x) = x_1 + x_2 - \frac{a}{2}\ln(x_2)$$

   and some number $c > 0$. Show that the set $S_c = \{x \in S \mid V(x) \leq c\}$ is invariant.

   Hint: Analyze the derivative of $V(x)$.

   (d) Show that all trajectories starting in S_c converge to the equilibrium point $\overline{x}$ using LaSalle's theorem. You may assume that the set S_c is compact.