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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2(a)	2(b)	2(c)	3(a)	3(b)	Exercise
3	4	4	4	5	5	25 Points

Consider the linear system

$$\ddot{y}(t) + 15\dot{y}(t) + 20y(t) = 25u(t). \tag{1}$$

1. Show that the transfer function of the system in (1) is $G(s) = \frac{25}{s^2+15s+20}$.

Next, consider the system in (1) connected to a controller $D(s)$ as shown in Figure 1.

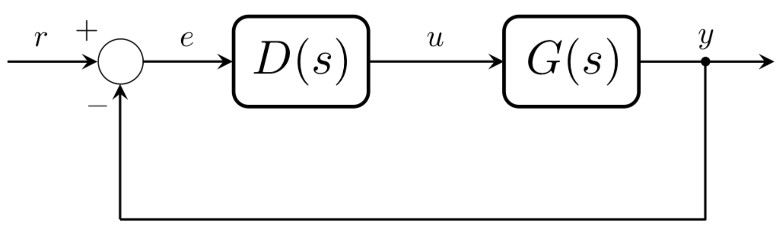

Figure 1: Closed-loop system

2. Consider first the controller $D(s) = K$, where K is a constant gain.

   (a) Compute the closed loop transfer function from the reference signal $r(t)$ to the tracking error $e(t)$.

   (b) Consider the reference signal $r(t) = 1$ for $t \geq 0$ (and $r(t) = 0$ for $t < 0$). Compute the range of K for which $\lim_{t\to\infty} e(t) < 0.1$.

   (c) Find the value of the K for which the closed-loop system has damping ratio $\zeta = 0.5$.

3. Consider now a dynamic controller

$$D(s) = K\frac{s + 1.5}{(s + 2)(s + 12)} \tag{2}$$

where again K is a constant gain. Figure 2 shows the Nyquist plot of $L(s) = G(s)D(s)$ for $K = 130$.

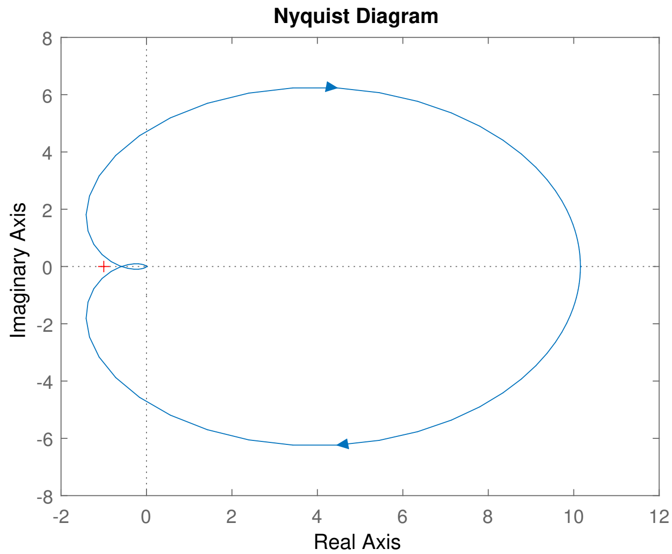

Figure 2: Nyquist plot of the open-loop transfer function $L(s) = G(s)D(s)$. The + shows the location of the point $(-1, 0)$.

(a) Will the closed-loop system be stable for this value of K?

(b) How will the stability of the closed-loop system change as K increases?

Exercise 2

1	2	3	4	5(a)	5(b)	5(c)	Exercise
3	6	4	4	2	3	3	25 Points

Consider the system

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 1 \\ \beta \end{bmatrix}}_{B} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} \gamma & 1 \end{bmatrix}}_{C} x(t)$$

where $x(t) \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ and $\beta, \gamma \in \mathbb{R}$ are constant parameters.

1. Is the system stable when $u(t) = 0$ for all t?
2. For which values of γ is the system observable? For which values of β is it controllable?
3. If you could choose between $\gamma = 0$ or $\gamma = 1$, which one would you choose and why?
   Hint: Recall the test rank $\begin{bmatrix} C \\ \lambda I - A \end{bmatrix}$ for detectability.
4. If you could choose between $\beta = 0$ or $\beta = -1$, which one would you choose and why?
5. Consider now the output feedback $u(t) = ky(t)$ where k is a constant. For $\beta = 2$ and $\gamma = 0$:
   (a) Write the closed loop system in the form $\dot{x}(t) = \tilde{A}(k)x(t)$, where $\tilde{A}(k)$ is a function of k.
   (b) For which values of k is the closed loop system asymptotically stable?
   (c) For which values of k does the system attain its fastest convergence rate.

Exercise 3

1	2	3(a)	3(b)	3(c)	Exercise
6	4	5	4	6	25 Points

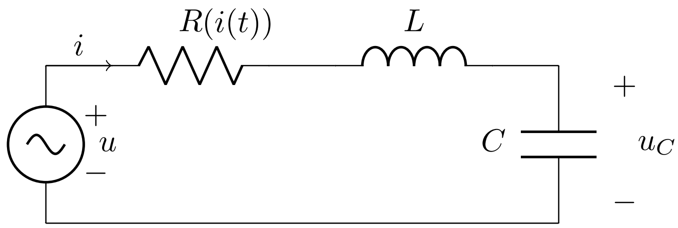

Figure 3: Basic RLC circuit.

Consider the RLC circuit depicted in Figure 3, where the adjustable voltage $u(t) \in \mathbb{R}$ is used to steer the voltage $u_c(t) \in \mathbb{R}$ and current $i(t) \in \mathbb{R}$. For simplicity assume that $C = 1F$ and $L = 1H$ and note that the resistance $R(\cdot) : \mathbb{R} \to \mathbb{R}$ is allowed to depend on the current that flows through it.

1. Derive a state-space model for the circuit in Figure 3 using $y(t) = i(t)$ as the output, $x(t) = \begin{bmatrix} u_c(t) \\ i(t) \end{bmatrix}$ as the state, and $u(t)$ as the input.

2. Under what conditions on the resistance function $R(i)$ is the system linear?

3. Consider now the resistance function $R(i) = -\frac{\sin(i)}{i}$ (where $R(0) = \lim_{i \to 0} R(i)$).

   (a) Find all equilibria of the system and determine their stability using linearisation.

   (b) Discuss the role of $R(i)$ using the power balance of the circuit when the magnitude of $i(t)$ is small. **Hint:** Recall that the power is an instantaneous energy change.

   (c) Assume that the output controller $u(t) = f(y(t)) + Ky(t)$ is applied to the system. Derive an expression for the function f and a range of values for K so that the closed loop system is linear and asymptotically stable.

Exercise 4

1	2	3(a)	3(b)	3(c)	Exercise
4	5	4	6	6	25 Points

Consider the optimization problem

$$\min_x \quad g(x) \tag{3}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a twice continuously differentiable function. Recall that the gradient vector, $\nabla g(x)$, and Hessian matrix, $\nabla^2 g(x)$ are defined as

$$\nabla g(x) = \begin{bmatrix} \frac{\partial g}{\partial x_1}(x) \\ \vdots \\ \frac{\partial g}{\partial x_n}(x) \end{bmatrix} \text{ and } \nabla^2 g(x) = \begin{bmatrix} \frac{\partial^2 g}{\partial^2 x_1}(x) & \dots & \frac{\partial^2 g}{\partial x_1 \partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g}{\partial x_1 \partial x_n}(x) & \dots & \frac{\partial^2 g}{\partial^2 x_1}(x) \end{bmatrix}.$$

Let $\overline{x} \in \mathbb{R}^n$ be a minimizer of g (that is $g(\overline{x}) = \min_x g(x)$) and recall that the gradient vanishes at the minimizer (that is $\nabla g(\overline{x}) = 0$). Finally, consider the dynamical system defined by

$$\dot{x}(t) = f(x(t)) = -\alpha \nabla g(x(t)), \tag{4}$$

where $\alpha > 0$ is a constant parameter.

1. Show that the minimizer $\overline{x}$ is an equilibrium of (4).

2. Assume that $\nabla^2 g(\overline{x}) \in \mathbb{R}^{n\times n}$ is a positive definite matrix. Show that $\overline{x}$ is locally asymptotically stable. **Hint:** Use Lyapunov's first method.

3. Consider the function $V(x) = \frac{1}{2}\|x - \overline{x}\|^2$.

   (a) Argue that $V(x) \geq 0$ for all $x \in \mathbb{R}^n$, $V(x) = 0$ if and only if $x = \overline{x}$, and $V(x) \to \infty$ whenever $\|x\| \to \infty$.

   (b) Show that $\frac{d}{dt}V(x(t)) = -\alpha \nabla g(x(t))^T(x(t) - \bar{x})$.

   (c) Assume that the gradient satisfies $(\nabla g(x) - \nabla g(y))^T(x - y) \geq m\|x - y\|^2$ for all $x, y \in \mathbb{R}^n$ and some $m > 0$. Show that in this case $\frac{d}{dt}V(x(t)) < 0$ for all $x \neq \overline{x}$. Hence argue that $\overline{x}$ is globally asymptotically stable using Lyapunov's second method.
   Hint: Recall that $\nabla g(\overline{x}) = 0$.