Automatic Control Laboratory
ETH Zürich
Prof. Florian Dörfler

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Signals and Systems II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4	5	6	7	Exercise
4	4	4	5	2	2	4	25 Points

Schwizerdütsch businessman Elön Müsk wants to hire a control theorist who knows how to land rockets, so that he can compete with the SBB–CFF–FFS on a new public transit initiative.
Your task is to model the rocket in Figure 1 with an engine at the bottom of the rocket, and two thrusters at the top of the rocket. The white–and–black circle denotes the center of gravity of the rocket.

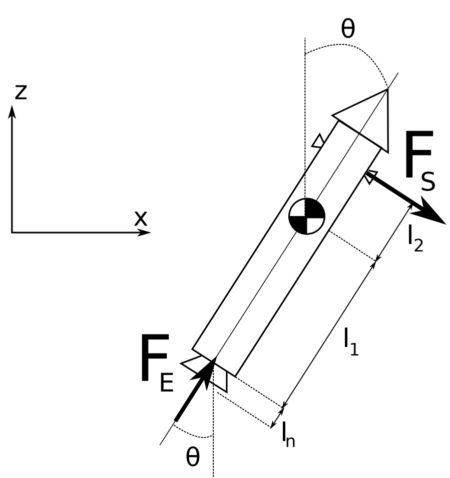

Figure 1: Elön Müsk's rocket. Only forces from the engine and thrusters are shown.

Quantities:

- m: Mass of the rocket
- J: Moment of inertia about the center of gravity of the rocket

- F_S: Net force from the two thrusters ($F_S > 0$ means that the left thruster is firing, $F_S < 0$ means that the right thruster is firing).
- F_E: Force from engine
- θ: Angle of rocket to z-axis
- l_1: Distance from the bottom of the rocket (top of the engine) to the center of mass of the rocket
- l_2: Distance from the center of mass of the rocket to the thrusters
- l_n: Length of the engine nozzle

1. Using the free-body diagram in Figure 1, write a differential equation for $\ddot{x}$.
   Hint: Use Newton's Third Law.
2. Using the free-body diagram in Figure 1, write a differential equation for $\ddot{z}$.
   Hint: Don't forget gravity.
3. Using the free-body diagram in Figure 1, write a differential equation for $\ddot{\theta}$. Assume that the force from the engine is applied to the base of the nozzle.
   Hint: Use the other Newton's Third Law, the one for rotational motion.
4. Notice that since rockets tend to point upwards, the angle θ is small. Convert the above equations for $\ddot{x}$, $\ddot{z}$, $\ddot{\theta}$ into a much simpler form using the **small angle approximation:**

$$\cos(\alpha) \approx 1, \;\; \sin(\alpha) \approx \alpha \quad \text{for small angles } \alpha$$

5. The inputs to the system are the thruster forces (F_E, F_S). Let $u = [F_E, F_S]^T$. Solve the approximate system in part 4 for the equilibrium input, i.e. the u such that $\ddot{x} = \ddot{z} = \ddot{\theta} = 0$ – solve this for the arbitrary equilibrium point x_e, z_e, θ_e. Does your solution actually allow θ_e to be arbitrary? What is the physical interpretation of this input? When is this interpretation valid?
6. Finally, by substituting $\dot{x} = v_x$, $\dot{z} = v_z$, $\dot{\theta} = v_\theta$, write everything neatly into a (possibly non-linear) state-space model of the form

$$\begin{bmatrix} \dot{x} \\ \dot{v_x} \\ \dot{z} \\ \dot{v_z} \\ \dot{\theta} \\ \dot{v_\theta} \end{bmatrix} = f(x, \dot{x}, z, \dot{z}, \theta, \dot{\theta}, F_E, F_S)$$

7. Assume that the rocket is in deep space, far away from the influence of any heavenly bodies. Elön Müsk has linearized the system about the origin, but forgot to compute

the B matrix. The A matrix is:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Compute the B matrix. Is the resulting system controllable?

(b) Suppose you have two sensors. Sensor 1 measures v_x, z, and θ. Sensor 2 measures x, z, and θ. Write down the correct C matrices for each sensor. Is the system observable using Sensor 1? Is the system observable using Sensor 2?

Exercise 2

1	2	3	4	5	Exercise
4	3	3	6	9	25 Points

Consider the LTI system

$$\begin{aligned} \dot{x}(t) &= \underbrace{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}}_{A} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$ and $a_1, a_2, a_3, a_4 \in \mathbb{R}$.

1. For which values of a_1, a_2, a_3, a_4 is system (1) controllable? For which values of a_1, a_2, a_3, a_4 is system (1) observable?

Assume now that $a_1 = 1$, $a_2 = 1$, $a_3 = 0$, $a_4 = 2$.

2. Find the set of points $[x_1, x_2]^\top \in \mathbb{R}^2$ that are observable.
3. Find the set of points $[x_1, x_2]^\top \in \mathbb{R}^2$ that can be reached at time $t = 1$ from initial condition $x(0) = [0, 0]^\top$ using a controller $u : [0, 1] \to \mathbb{R}$, i.e., find the set

$$X = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; \exists\, u : [0,1] \to \mathbb{R} \text{ s.t. } x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } x(1) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\}.$$

4. Find the matrix exponential e^{At} for $t \in \mathbb{R}$.
5. Is it possible to find a controller $u : [0, 1] \to \mathbb{R}$ that steers system (1) from $x(0) = [0, 0]^\top$ to $x(1) = [1, 1]^\top$? If so, find a controller that does this. If not, justify your reason.

Exercise 3

1	2	3	4	5	6	Exercise
4	5	4	4	5	3	25 Points

Consider the following dynamical system

$$\dot{x}(t) = rx(t) - x^3(t), \tag{2}$$

where $x(t) \in \mathbb{R}$ is the state variable and $r \in \mathbb{R}$ is a scalar parameter.

1. Determine all the equilibria of system (2) for $r < 0$, $r = 0$ and $r > 0$.

2. Analyse the stability of all found equilibria with the linearization method (Lyapunov's indirect method).

3. Analyse the stability of the equilibrium associated to $r = 0$ with Lyapunov's direct method.
   Hint: Use a quadratic candidate Lyapunov function.

4. Show that the stable equilibrium associated to $r < 0$ is globally asymptotically stable. Can you make the same conclusion for the stable equilibrium associated to $r = 0$? And for $r > 0$?

5. Depict, in the $r - x$ plane, all found equilibria as a function of the parameter r. Draw the stable equilibria with solid lines, and the unstable equilibria with dashed lines. Finally, sketch some sample trajectories to show qualitatively the behaviour of system (2) in each sector $r < 0$, $r = 0$ and $r > 0$.

6. The plot you obtained in the previous task is called *bifurcation diagram.* Now link the following three dynamical systems to the corresponding bifurcation diagram in Fig. 2, by ticking the correct boxes below.

   - $\dot{x}(t) = rx(t) + x^3(t)$ | a | b | c | d | none |

   - $\dot{x}(t) = -rx(t) + x^3(t)$ | a | b | c | d | none |

   - $\dot{x}(t) = -rx(t) - x^3(t)$ | a | b | c | d | none |

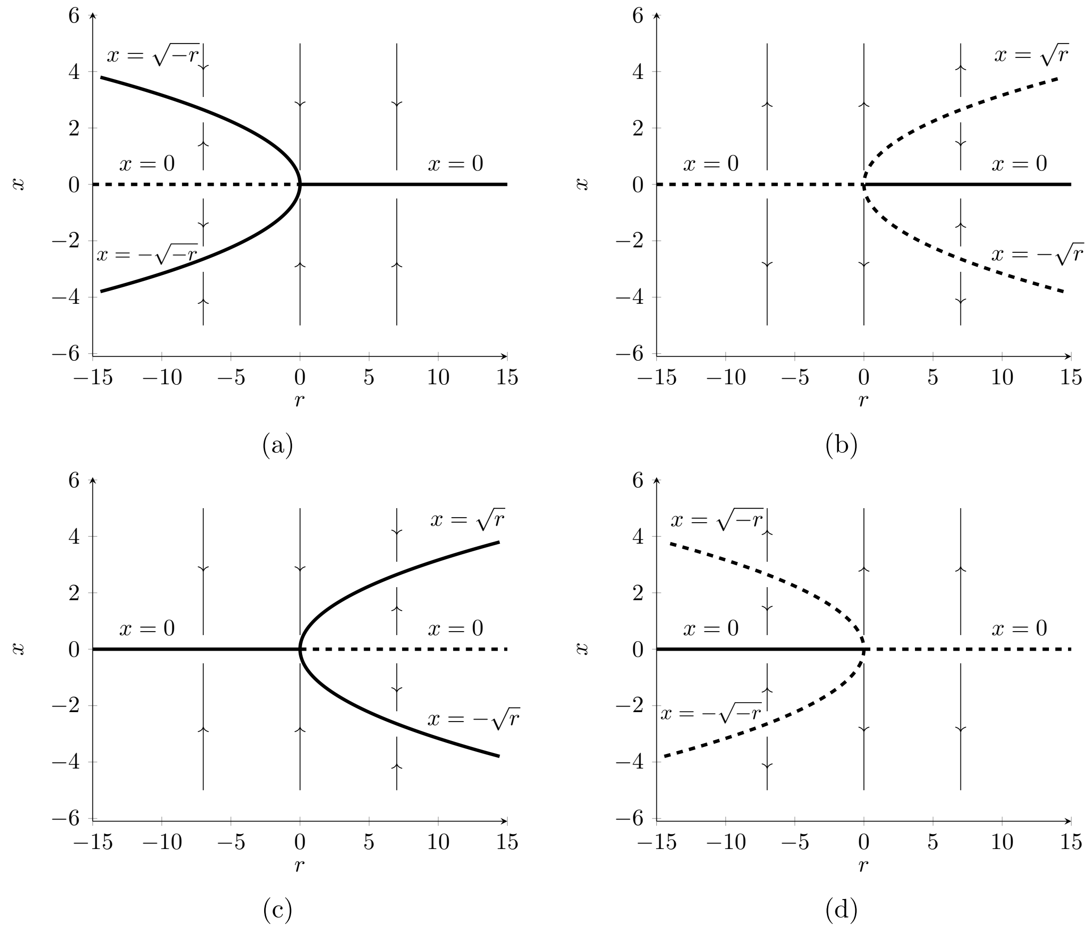

Figure 2: Bifurcation diagrams. The solid lines are stable equilibria and the dashed lines are unstable equilibria.

Exercise 4

1	2	3	4	5	Exercise
3	6	6	4	6	25 Points

Consider the linear system

$$\Sigma_1 : \quad \begin{aligned} \dot{x}_1(t) &= A\,x_1(t) + B\,u_1(t) \\ y_1(t) &= C\,x_1(t) + D\,u_1(t) \end{aligned}$$

given by the matrices

$$A = \begin{bmatrix} -2 & -5 \\ 1 & a \end{bmatrix} \ , \ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ , \ C = \begin{bmatrix} 0 & 1 \end{bmatrix} \ , \ D = 0$$

where $a \in \mathbb{R}$ is a system parameter.

1. Compute the transfer function $G_1(s)$ of the system Σ_1.

2. Based only on your answer in Part 1:
   (a) Determine the values of the parameter a for which the system Σ_1 is critically damped.
   Hint: A second order system is critically damped when its poles are real and equal.
   (b) Find the natural frequency of the linear system Σ_1 in terms of the parameter a.
   (c) Determine the value of a for which $y_1(t)$ is unbounded for $u_1(t) = 3\sin(t)$.

Next, consider the system Σ_1 connected to a controller Σ_2 as shown in Figure 3.

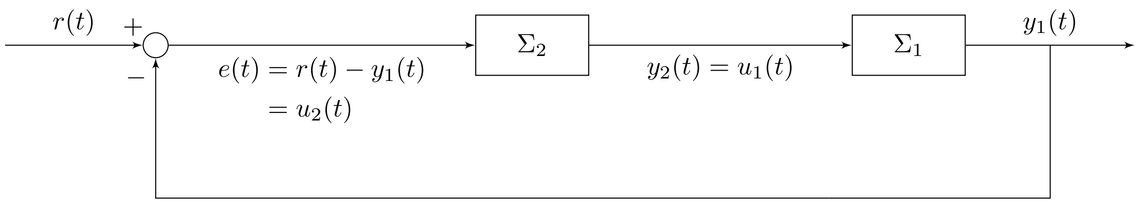

Figure 3: Feedback system

3. Consider the controller $\Sigma_2 : y_2(t) = Ke(t)$ and let $a = -10$.
   (a) Compute the closed loop transfer function.
   (b) For which values of K can one guarantee stability of the closed loop shown in Figure 3?

Now, consider an LTI system with the transfer function $F(s)$ with input signal $u(t)$ and output signal $y(t)$, i.e. $Y(s) = F(s)U(s)$. Corresponding to

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

the zero-state response $y_0(t)$ is a ramp signal,

$$y_0(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$

4. Find the Laplace transform of $y_0(t)$ and the transfer function $F(s)$.
5. Find the Laplace transform of

$$y_b(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2 - t & 1 < t \leq 2 \\ 0 & t > 2 \end{cases}$$

   Hint: You can express $y_b(t)$ as a sum of time-shifted ramp signals $y_0(t)$. Also, $L\{y_0(t-c)\} = e^{-cs}Y_0(s)$ for $c > 0$, where L denotes the Laplace transform.