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D-ITET
Winter 2018/2019
7.02.2019

Signals and Systems II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	Exercise
8	9	8	25 Points

Consider the discrete time linear system

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$.

1. A discrete time observer constructs an estimate $\hat{x}_k$ of the state x_k of the form

$$\begin{cases} \hat{x}_{k+1} &= A\hat{x}_k + Bu_k + L(y_k - \hat{y}_k) \\ \hat{y}_k &= C\hat{x}_k + Du_k \end{cases}$$

   starting with $\hat{x}_0 = 0$. Derive the dynamics of the error $e_k = x_k - \hat{x}_k$. Under what conditions will $\hat{x}_k$ converge to x_k as k tends to infinity?

2. Consider now the system (1) with

$$A = \begin{bmatrix} 0 & 0 & -9 \\ 1 & 0 & -15 \\ 0 & 1 & -7 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2}$$

$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \qquad D = 0$$

   Is the system observable? Is it controllable? Show that the open-loop system has poles at $-1 + j0$, $-3 + j0$, and $-3 + j0$. What does this mean for the stability of the system?

3. Design an observer of the form given in part 1 for the system in part 2. What dimensions should the matrix L have? Select the entries of the matrix L so that the error dynamics have all eigenvalues equal to 0.5.

Exercise 2

1	2	3	4	5	Exercise
5	4	4	8	4	25 Points

Figure 1 shows a water storage system consisting of a cylindrical tank which has constant surface area A_t. Water flows into the tank at a fixed mass-flow rate q_{in}. The mass-flow rate at which the water flows out of the tank can be controlled with a valve whose outflow is given by

$$q_{\text{out}}(t) = \rho A_v \sqrt{2gh(t)}\, v(t),$$

where $h(t)$ is the height of water in the tank and

ρ	–	water density,
A_v	–	cross section area of the fully open valve,
g	–	acceleration due to gravity

are constants. The valve position $v(t) \in [0, 1]$ can be controlled with a servo motor with the transfer function

$$G(s) = \frac{V(s)}{U(s)} = \frac{2}{s+30},$$

where $u(t)$ is the input voltage to the motor.

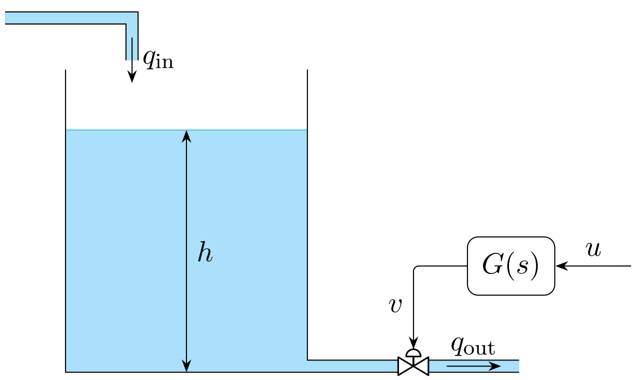

Figure 1: Water storage system.

1. Show that the system dynamics takes the form

$$\dot{h}(t) = f_1(h(t), v(t), u(t)) = \frac{q_{\text{in}}}{\rho A_t} - \frac{A_v}{A_t}\sqrt{2gh(t)}\, v(t),$$
$$\dot{v}(t) = f_2(h(t), v(t), u(t)) = -30v(t) + 2u(t).$$

2. Explain whether the system above is
   (a) linear or nonlinear,
   (b) time-variant or time-invariant.

3. Compute the valve position $\bar{v}$ that maintains the water-level at a constant height $\bar{h}$. What is the input voltage $\bar{u}$ in that case?

4. Linearize the system around the steady state computed in part 3, *i.e.* compute the matrices A and B in

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix}\right) = \underbrace{\begin{bmatrix}\frac{\partial f_1(h,v,u)}{\partial h} & \frac{\partial f_1(h,v,u)}{\partial v}\\ \frac{\partial f_2(h,v,u)}{\partial h} & \frac{\partial f_2(h,v,u)}{\partial v}\end{bmatrix}\Bigg|_{\substack{h=\bar{h}\\ v=\bar{v}\\ u=\bar{u}}}}_{A}\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} + \underbrace{\begin{bmatrix}\frac{\partial f_1(h,v,u)}{\partial u}\\ \frac{\partial f_2(h,v,u)}{\partial u}\end{bmatrix}\Bigg|_{\substack{h=\bar{h}\\ v=\bar{v}\\ u=\bar{u}}}}_{B}\delta u(t),$$

   where $\delta h(t) := h(t) - \bar{h}$, $\delta v(t) := v(t) - \bar{v}$, and $\delta u(t) := u(t) - \bar{u}$.

5. Is the linearized system stable? What can you infer about the stability of the non-linear system?

Exercise 3

1	2	3	4	5	Exercise
3	3	7	8	4	25 Points

Consider the following nonlinear differential equation:

$$\ddot{y}(t) + c\dot{y}(t) + \sin(y(t)) = 0, \tag{3}$$

where $y(t) \in \mathbb{R}$ for $t \geq 0$, and $c \in \mathbb{R}$.

1. By defining states $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$, write equation (3) in the form

$$\dot{x}(t) = f(x(t)),$$

   where $x(t) = (x_1(t), x_2(t))$.

2. Find all equilibrium points for the system above.

3. Determine the stability of the equilibrium point $x = (0, 0)$ for $c < 0$ and $c > 0$ using linearization.

4. Can you determine the stability of the equilibrium point $x = (0, 0)$ when $c = 0$ using linearization? If not, find a suitable Lyapunov function to show that the equilibrium is stable. [Hint: Think pendulum]

5. Your friend from EPFL thinks that the equilibrium point $x = (0, 0)$ is globally asymptotically stable for $c = 1$. Explain why this is not the case.

Exercise 4

1	2	3	Exercise
5	10	10	25 Points

Consider the linear system Σ given by:

$$\Sigma : \qquad \ddot{y}(t) + (2 - \alpha)\dot{y}(t) - 2\alpha y(t) = \dot{u}(t) - u(t) \tag{4}$$

where $\alpha \in \mathbb{R}$ is a system parameter.

1. Compute the transfer function $G(s)$ of the system Σ.

Next, consider the system Σ connected to a controller as shown in Figure 2.

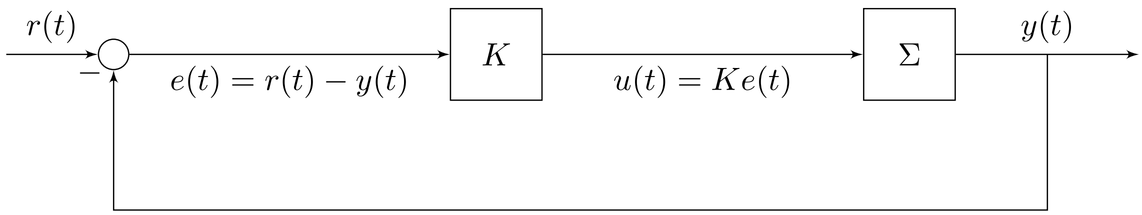

Figure 2: Feedback system

2. Consider the case $\alpha = -0.01$. Figure 3 shows the Bode diagram of the transfer function $G(s)$ of the system Σ for $\alpha = -0.01$. The maximum of the gain $|G(j\omega)|$ occurs at $\omega = 0$.

   (a) Compute $|G(j\omega)|$ at $\omega = 0$.

   (b) Apply the Small gain theorem (corollary of the Bode Stability criterion) to obtain a range for the control gain K for which the feedback system shown in Figure 2 is stable.

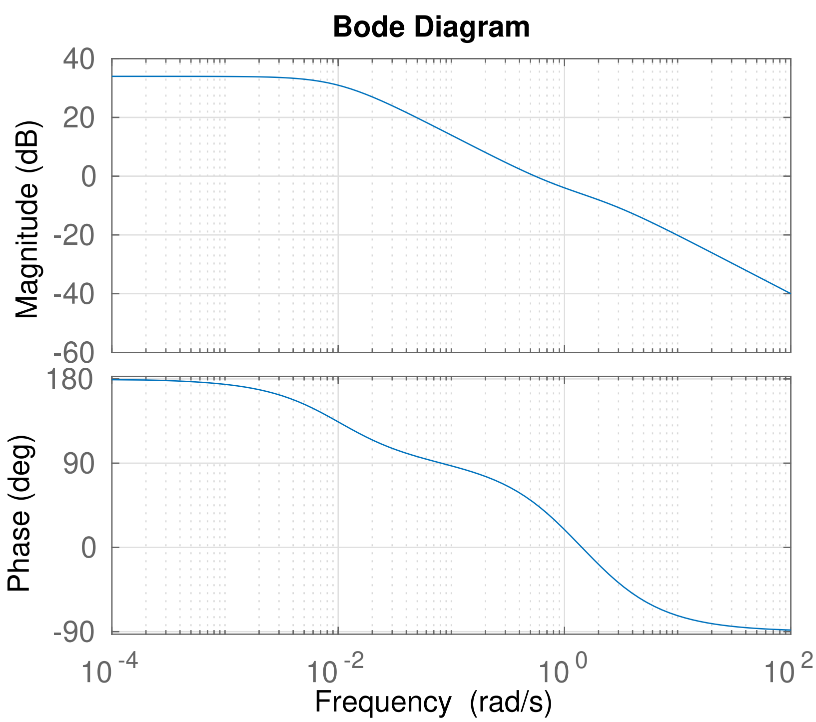

Figure 3: Bode diagram of the open-loop transfer function of the system Σ for $\alpha = -0.01$.

3. Next, consider the case $\alpha = 10$. Figure 4 shows the Nyquist diagram of the transfer function $G(s)$ of the system Σ for $\alpha = 10$. Use the Nyquist stability criterion to show that there does not exist a K for which the closed loop system is stable.

   [Hint: Show that the Nyquist criterion is violated both when $\frac{1}{K}$ is encircled by the Nyquist plot and when it is not.]

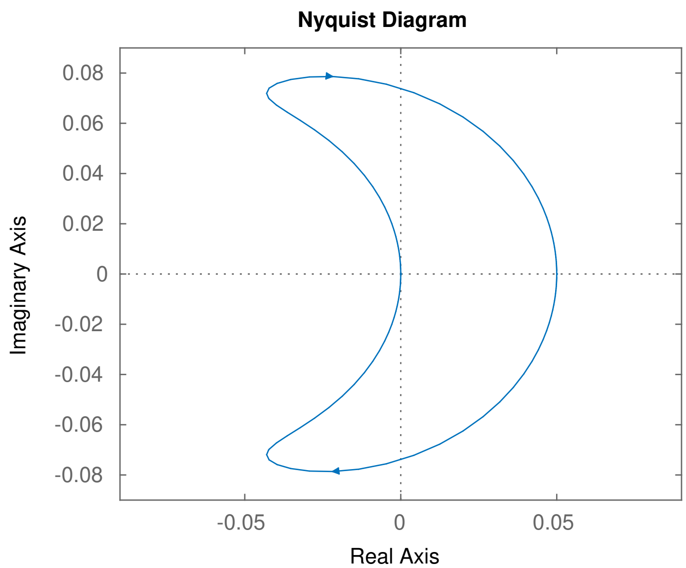

Figure 4: Nyquist diagram of the open-loop transfer function $G(s)$ of the system Σ for $\alpha = 10$.