Automatic Control Laboratory
ETH Zurich
Prof. J. Lygeros

D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4	Exercise
6	6	8	5	25 Points

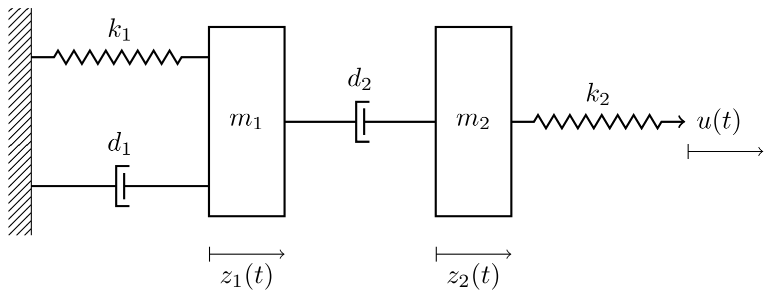

Figure 1: Mechanical system

Consider the spring-mass-damper system shown in Figure 1. Assume that $u(t)$ denotes the position of the right end of spring k_2 and that all positions $z_1(t), z_2(t), u(t)$ are measured with respect to the equilibrium state where all forces are zero.

1. Using $x(t) = \begin{bmatrix} z_1(t) & \dot{z}_1(t) & z_2(t) & \dot{z}_2(t) \end{bmatrix}^T$ as the state vector, $u(t)$ as input and $y(t) = z_2(t)$ as output, determine the matrices A, B, C, D of the state-space representation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

2. Under what conditions is the system observable? You may assume that all parameters have positive values.

3. Assume now that all parameters are positive, but damper d_2 breaks, meaning $d_2 = 0$. Compute the eigenvalues of the system. Which of these “modes” will be controllable from the input $u(t)$? Which will be observable from the output $y(t)$? Is the system stabilizable? Is it detectable?

4. Your friend from EPFL is convinced that he can find a change of coordinates $\hat{x}(t) = Tx(t)$ to restore the observability of the system of part 3. He is now busy looking for the right $T \in \mathbb{R}^{n \times n}$. Prove to him mathematically that he is wasting his time.

Exercise 2

1	2	3	4	Exercise
5	8	4	8	25 Points

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

given by the matrices

$$A = \begin{bmatrix} \sigma & \omega_0 \\ -\omega_0 & \sigma \end{bmatrix} \ , \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ , \ C = \begin{bmatrix} 1 & 0 \end{bmatrix} \ , \ D = 0$$

in which $\omega_0 \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and $\omega_0 > 0$, $|\omega_0| > |\sigma|$.

1. Compute the transfer function $G(s)$. What are the poles of the system?

2. Show that the magnitude $|G(j\omega)|$ as a function of ω has a maximum at $\omega = \sqrt{\omega_0^2 - \sigma^2}$.

3. Compute the value of the maximum, that is, compute $|G(j\sqrt{\omega_0^2 - \sigma^2})|$.

4. Figure 2 depicts the impulse responses and the Bode plots of three linear systems of the form studied in this exercise, for parameter values

   (i) $\sigma = -0.2$, $\omega_0 = 10$

   (ii) $\sigma = -1$, $\omega_0 = 20$

   (iii) $\sigma = -1$, $\omega_0 = 10$

   in a random order. Pair the figures of the impulse responses and the bode plots with the corresponding parameter values and state a *brief* reason for each pairing. (Hint: Your solutions to the previous three questions might be helpful.)

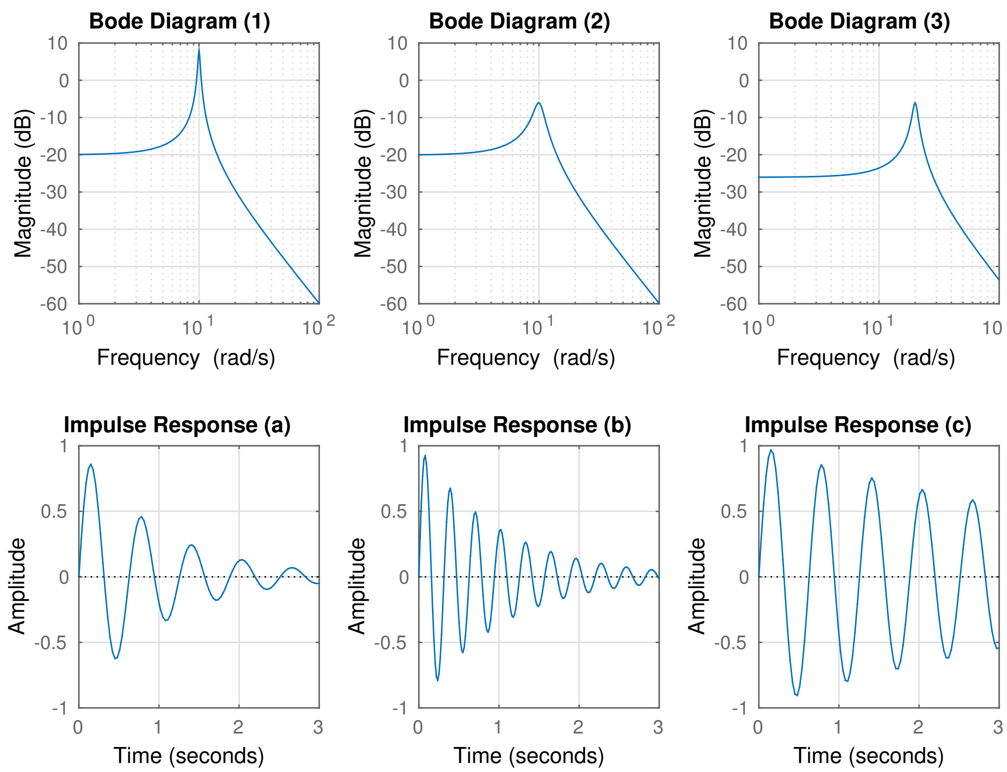

Figure 2: Bode plots and impulse responses.

Exercise 3

1	2	3	4	5	Exercise
2	4	6	6	7	25 Points

Consider the system described by the ODE

$$\frac{d^2}{dt^2}y(t) + \theta\frac{d}{dt}y(t) + \left(e^{-t} + 1\right)y(t) = 0, \tag{1}$$

where $\theta \in \mathbb{R}$ is a constant parameter.

1. Is the system (1) linear? Is it time invariant?

2. Using the states $x_1(t) = y(t)$, $x_2(t) = \frac{d}{dt}y(t)$, $x_3(t) = e^{-t/2}$ write the system (1) in time invariant state space form. What initial condition must be selected for $x_3(0)$ for the time varying form (1) and the time invariant form to be equivalent? Is the time invariant state space form linear?

3. What are the equilibria of the system derived in part 2? Whenever possible, determine the stability of the equilibria for different values of θ using linearization.

4. Assume $\theta > 0$ and consider the function $V(x) = x_1^2 + \frac{x_2^2}{x_3^2+1} + \alpha x_3^2$, where $\alpha > 0$ is a positive constant. Compute its Lie-derivative along the system dynamics (i.e., $\frac{d}{dt}V(x(t))$). Determine conditions on $x(t) \in \mathbb{R}^3$ such that the Lie-derivative is less than or equal to zero.

5. Can you conclude anything about the stability of the equilibria in part 3 from your calculations in part 4?

Exercise 4

1	2	3	4	5	Exercise
3	6	4	6	6	25 Points

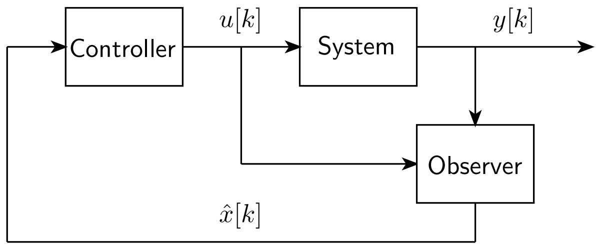

Figure 3: Closed loop system comprised of system, observer and controller.

The engineers at "ACHME Space Products LTD", where you are working as an intern, have designed a mechanical system for their latest satellite. They modelled the dynamics of its orientation by the discrete-time linear system

$$x[k+1] = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k].$$

1. Is the system open-loop (i.e. when $u[k] = 0$ for all $k \geq 0$) stable?

2. Assuming that the whole state $x[k]$ can be measured, you designed a feedback controller $u[k] = -Kx[k]$ with $K = \begin{bmatrix} \frac{9}{4} & 2 \end{bmatrix}$. Is the closed loop using your controller stable?

3. The ACHME engineers inform you that it will unfortunately not be possible to measure the two states, $x_1[k]$ and $x_2[k]$, independently. They offer you a choice: Either a sensor that measures only $x_1[k]$ or a sensor that measures $x_1[k] + x_2[k]$. Which of the two sensors will you pick?

4. With your choice of sensor from part 3, the output of the system becomes $y[k] = Cx[k]$ for an appropriate matrix C. You would like to implement an observer of the form
$$\hat{x}[k+1] = A\hat{x}[k] + Bu[k] + L(y[k] - \hat{y}[k]),$$
with $\hat{y}[k] = C\hat{x}[k]$, to generate an estimate, $\hat{x}[k]$, of the state from measurements of $u[k]$ and $y[k]$. Derive the equation for the dynamics of the estimation error $e[k] = x[k] - \hat{x}[k]$.

5. Compute the observer gain L that will place the eigenvalues of the estimation error dynamics at $\frac{1}{4}, \frac{1}{4}$. What would you expect to happen if you connect the observer, the system and your controller $u[k] = -K\hat{x}[k]$ from part 2 in closed loop as shown in Figure 3?