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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4	Exercise
8	6	5	6	25 Points

Consider the following circuit:

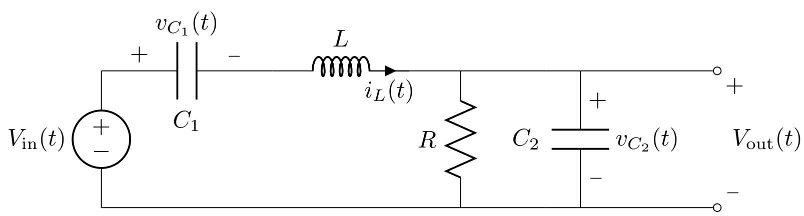

Figure 1: Electrical circuit

1. Using $x(t) = \begin{bmatrix} i_L(t) & v_{C_1}(t) & v_{C_2}(t) \end{bmatrix}^T$ as state vector, $u(t) = V_{\text{in}}(t)$ as input and $y(t) = V_{\text{out}}(t)$ as output, derive the state space description of the circuit in the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{1}$$

2. Assume that $R = C_1 = C_2 = L = 1$. Design an observer $K \in \mathbb{R}^3$ for the system such that the observer error dynamics

$$\dot{e}(t) = (A - KC)e(t) \tag{2}$$

have all eigenvalues at -1.

Consider now an LTI system given by

$$\begin{aligned} \dot{x}(t) &= \underbrace{\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{B} u(t) \\ y(t) &= \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{C} x(t). \end{aligned} \tag{3}$$

3. Determine whether the system (3) is controllable and observable.
4. Find all eigenvalues of A in (3). Is the system stable? Determine whether the system is detectable. (HINT: A system is detectable if $\begin{bmatrix} C \\ \lambda_i I - A \end{bmatrix}$ has full rank for all non-negative eigenvalues λ_i of A.)

Exercise 2

1	2	3	4	5	Exercise
6	5	2	5	7	25 Points

1. Consider an autonomous system:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}}_{A} x(t) \ , \tag{4}$$

   where σ and ω are two real numbers. Determine the eigenvalues and eigenvectors of A.

2. For the system described by (4), verify that the state transition matrix is given by

$$\Phi(t, t_0) = \exp(\sigma t) \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} \ . \tag{5}$$

3. For what values of σ and ω is the system in (4) stable? For what values is the system asymptotically stable?

4. Now consider the system

$$\dot{x}(t) = \underbrace{\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{B} u(t) \ . \tag{6}$$

   For which combinations of σ, ω, b_1 and b_2 is this system controllable? For which combinations is it not?

5. Next, consider the closed-loop system of (6) with $B = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ under the state feedback law $u(t) = Kx(t)$, with $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. By appropriately selecting K, one can change the eigenvalues of the closed-loop system $\dot{x}(t) = (A+BK)x(t)$. What are the possible eigenvalues one can obtain when $\omega \neq 0$ and when $\omega = 0$?

Exercise 3

1	2	3	4	5	Exercise
3	2	5	7	8	25 Points

Consider the system

$$\begin{aligned}\dot{x}(t) &= -x(t) + g(y(t)), \\ \dot{y}(t) &= -y(t) + h(x(t)),\end{aligned} \tag{7}$$

where $x(t), y(t) \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ are two unknown continuously differentiable functions satisfying

$$|g(y)| \leq \frac{|y|}{2} \quad \forall y \in \mathbb{R}, \quad |h(x)| \leq \frac{|x|}{2} \quad \forall x \in \mathbb{R}.$$

1. Is system (7) linear? Is it autonomous? Is it time invariant?
2. Show that $(x, y) = (0, 0)$ is an equilibrium point of system (7).
3. Prove the following properties
   (a) $|\frac{d}{dx}(h(x))_{x=0}| \leq \frac{1}{2}$ and similarly $|\frac{d}{dy}(g(y))_{y=0}| \leq \frac{1}{2}$.
   (b) $|xy| \leq \frac{x^2+y^2}{2}$ for all $x, y \in \mathbb{R}$.
4. Determine, if possible, the stability properties of the equilibrium $(x, y) = (0, 0)$, using the linearization technique.
   (HINT: Use property 3(a).)
5. Show that $V(x, y) = \frac{x^2+y^2}{2}$ is a Lyapunov function for system (7). Can you use this information to find out more about the stability properties of the equilibrium $(x, y) = (0, 0)$ compared to your answer in part 4?
   (HINT: Use property 3(b).)

Exercise 4

1	2	3	4	5	6	Exercise
4	4	8	3	3	3	25 Points

Consider the discrete time system given by the difference equation:

$$y(k) - a_1 y(k-1) - a_2 y(k-2) = b_0 u(k) + b_1 u(k-1) + b_2 u(k-2) \tag{8}$$

Digital filters are often defined in this way. One way to implement such a filter is a feedforward-feedback structure as depicted in Figure 2:

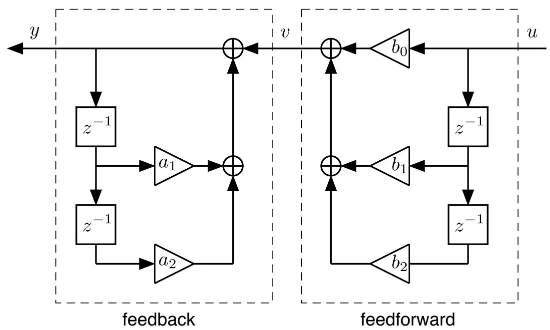

Figure 2: Digital filter

For simplicity, we consider the case $a_1 = 0$. The state-space realization of the feedforward stage is given as:

$$z(k+1) = A_1 z(k) + B_1 u(k) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \tag{9a}$$

$$v(k) = C_1 z(k) + D_1 u(k) = \begin{bmatrix} b_2 & b_1 \end{bmatrix} z(k) + b_0 u(k) \tag{9b}$$

and the one of the feedback stage as:

$$x(k+1) = A_2 x(k) + B_2 v(k) = \begin{bmatrix} 0 & a_2 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v(k) \tag{10a}$$

$$y(k) = C_2 x(k) + D_2 v(k) = \begin{bmatrix} 0 & a_2 \end{bmatrix} x(k) + v(k) \tag{10b}$$

First, we analyze the feedforward stage and the feedback stage separately:

1. For which parameter values b_0, b_1 and b_2 is the feedforward stage (9) asymptotically stable? For which parameter values is it observable?

2. For which parameter values a_2 is the feedback stage (10) asymptotically stable? For which parameter values is it observable?

Now, we will analyze the complete system:

3. Using the state $\xi(k) := \begin{bmatrix} x(k)^\top & z(k)^\top \end{bmatrix}^\top$, find the matrices A, B, C and D that describe the complete system as:

$$\xi(k+1) = A\xi(k) + Bu(k) \tag{11a}$$
$$y(k) = C\xi(k) + Du(k) \tag{11b}$$

4. For which parameter values a_2, b_0, b_1 and b_2 is the complete system (11) asymptotically stable?

5. What is the minimal number of states necessary to implement the system given by (8)? Justify your answer. (HINT: Compute the z-transformation of Equation (8) and find the transfer function $G(z)$.)

6. Assume that the parameter values are given as $a_2 = b_0 = b_1 = b_2 = 1$. You may use the fact that for this parameter choice, the complete system (11) is controllable. Is the complete system observable? (HINT: The answer to question 5 might be useful for this question.)