Automatic Control Laboratory
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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4	5	Exercise
8	4	2	7	4	25 Points

Consider the following circuit:

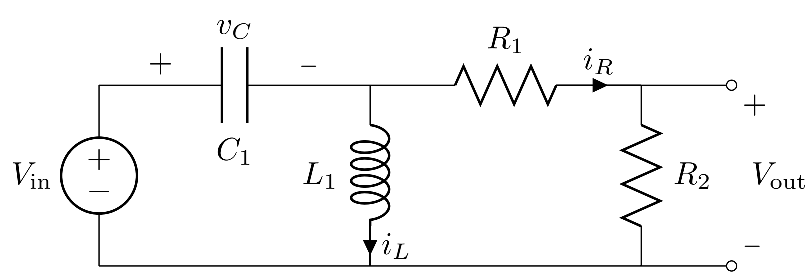

Figure 1: Electrical circuit

1. Using $x(t) = \begin{bmatrix} v_C(t) & i_L(t) \end{bmatrix}^T$ as state vector, $u(t) = V_{\text{in}}(t)$ as input and $y(t) = V_{\text{out}}(t)$ as output, derive the state space description of the given circuit in the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{1}$$

2. Let now $C_1 = 1\text{F}, L_1 = 1\text{H}$ and $R_1 = R_2 = 2\Omega$. Show that the system is both controllable and observable. In case you have not been able to solve part 1, continue with the following matrices:

$$A = \begin{bmatrix} -\frac{1}{4} & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix}, \quad D = \frac{1}{2} \tag{2}$$

3. Assume (for this part only) something went wrong in the factory and the circuit from Fig. 1 was shipped with $R_2 = 0\Omega$. Is the circuit still observable with this fault? Make an intuitive argument. Is the broken circuit detectable?

4. Calculate the transfer function $G(s)$ from V_{in} to V_{out} given the system matrices in (2).

5. Your friend from EPFL experimented with the circuit using voltages of the form

$$V_{\text{in}}(t) = \cos(\omega_0 t) + V_0 \tag{3}$$

and noticed that the steady state behavior of $V_{\text{out}}(t)$ did not depend on V_0. Explain either physically or mathematically why this is.

Exercise 2

1	2	3	4	5	Exercise
3	5	8	5	4	25 Points

Consider the following transfer function of a linear time-invariant system:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\omega^2}{s^2 + 2\omega\zeta s + \omega^2}, \tag{4}$$

The parameters $\omega \geq 0$ and ζ are called the natural frequency and the damping factor of the system, respectively.

1. Under what conditions on ω and ζ is the system (4) asymptotically stable?
2. From (4), derive a differential equation that describes the output trajectory $y(t)$ for a given input signal $u(t)$. Assume $y(0) = \dot{y}(0) = \ddot{y}(0) = 0$.
3. Use $x_1(t) := y(t)$ and $x_2(t) := \dot{y}(t)$ to write down the matrices A, B, C, and D of a state space realization of system (4). Can there be other second-order realizations of (4)? Can there be realizations of (4) that are third-order or higher?

For the remainder of the exercise, assume the following state space realization of (4):

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\omega\zeta \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} \omega^2 & 0 \end{bmatrix}, \qquad D = 0 \tag{5}$$

4. A state-feedback controller $u(t) = Kx(t)$ with $K \in \mathbb{R}^{1\times 2}$ should be designed to obtain the closed-loop system

   $$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & a \end{bmatrix} x(t), \quad y(t) = \begin{bmatrix} \omega^2 & 0 \end{bmatrix} x(t), \tag{6}$$

   with $a \leq -5$. Provide values for the entries of the state feedback matrix K that will guarantee the desired system structure in (6).
5. Is it possible to design a state feedback controller $u(t) = Kx(t)$ for the realization (5) that places the poles of the closed-loop system at -1 and -2?

Exercise 3

1	2	3	4	Exercise
3	4	8	10	25 Points

The Lotka-Volterra equations are frequently used to describe the dynamics of biological systems with two species, one predator and one prey. The populations change through time according to

$$\begin{aligned} \dot{x}(t) &= x(t) - x(t)y(t), \\ \dot{y}(t) &= x(t)y(t) - y(t), \end{aligned} \tag{7}$$

where $x(t) \geq 0$ is the number of prey individuals at time t and $y(t) \geq 0$ is the number of predator individuals at time t.

1. Is system (7) linear? Is it autonomous? Is it time invariant?
2. Compute the equilibrium points of system (7).
3. Determine, whenever possible, the stability properties of the equilibria found in part 2 using the linearization technique.
4. Consider the function $V(x, y) = -xye^{-(x+y)} + e^{-2}$.
   (a) Show that $V(x, y) = 0$ in one (and only one) of the equilibrium points found in part 2. Denote this point by $(\bar{x}, \bar{y})$.
   (b) The level sets of $V(x, y)$ around $(\bar{x}, \bar{y})$ are given in Figure 2. Assume that these level sets are compact and $V(x, y) > 0$ locally around $(\bar{x}, \bar{y})$. Compute the Lie derivative (derivative along the trajectories) of $V(x, y)$. What can you say about the stability properties of $(\bar{x}, \bar{y})$?
   (c) Based on Figure 2 and on the Lie derivative of $V(x, y)$, sketch the trajectories of the system for some initial conditions in the x-y positive orthant.
   HINT: Compute the sign of the derivative $\dot{x}$ and $\dot{y}$, when $x < \bar{x}$ or $x > \bar{x}$ and when $y < \bar{y}$ or $y > \bar{y}$.

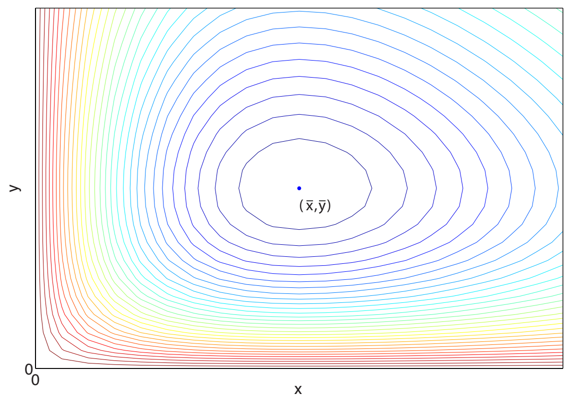

Figure 2: Level sets of the function $V(x, y)$ in Exercise 2.

Exercise 4

1	2	3	4	Exercise
3	11	8	3	Points

Consider the following discrete–time system:

$$x(k+1) = Ax(k) + Bu(k) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k), \tag{8}$$

where $x \in \mathbb{R}^3$, $u \in \mathbb{R}$ and the parameters $a_1, a_2, a_3 \in \mathbb{R}$.

1. For which parameter values a_1, a_2 and a_3 is the system controllable?
2. Suppose that you want to control the system using linear state–feedback, i.e. $u(k) = Kx(k)$, $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \in \mathbb{R}^{1\times 3}$.
   (a) Derive the system matrix A_K of the closed–loop system $x(k+1) = A_K x(k)$.
   (b) Compute the characteristic polynomial of the closed–loop system.
   (c) We want to use pole-placement to determine the controller parameters. Find controller parameters k_1, k_2 and k_3 such that the closed–loop system matrix has eigenvalues λ_1, λ_2 and λ_3. Is this possible for any $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$?
3. For system (8), we want to design a so–called *deadbeat controller*, that is, linear state–feedback such that the closed–loop system converges to the origin in a finite number of steps, for any initial state $x(0) = x_0 \in \mathbb{R}^3$. Find $k_1, k_2, k_3 \in \mathbb{R}$ such that the linear feedback controller $u(k) = Kx(k)$ with $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$ is deadbeat. HINT: Recall that a matrix is nilpotent if (and only if) its eigenvalues are all equal to zero.
4. Can we find a linear deadbeat controller for a controllable, linear system in continuous time?