Automatic Control Laboratory
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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	**2**	**3**	**4**	**5**	**Exercise**
5	**4**	**3**	**8**	**5**	**25 Points**

The following schematics show two simple RC circuits, to be used as high-pass and low-pass filters. Assume that in both cases the voltage across the capacitor is initially zero.

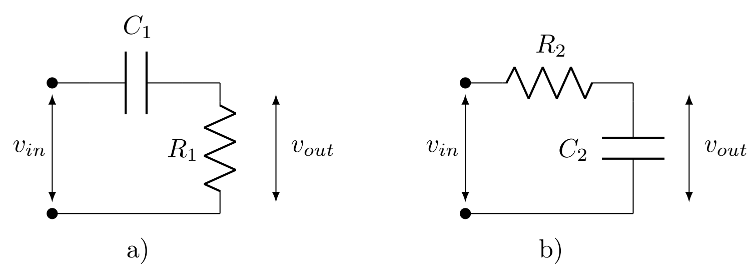

Figure 1: a) high-pass filter and b) low-pass filter

1. Write the dynamics of the high-pass filter in Figure 1 a). Then, use the Laplace transform to compute the transfer function $G_{\text{HP}}(s)$ from the input v_{in} to the output v_{out}.
2. Repeat the exercise for the low pass filter in Figure 1 b) and derive the transfer function $G_{\text{LP}}(s)$.
3. A band-pass filter is a series combination of a high-pass filter followed by a low-pass filter. Draw the block diagram of a band-pass filter using the high- and low-pass blocks. Using the transfer functions from Parts 1 and 2 verify that the transfer function of the band-pass filter is
$$G_{\text{BP}}(s) = \frac{sR_1C_1}{(sR_1C_1+1)(sR_2C_2+1)}.$$
4. For $R_1 = 0.4\text{k}\Omega$, $C_1 = \frac{1}{2\pi}\mu\text{F}$, $R_2 = \frac{1}{\pi}\text{k}\Omega$ and $C_2 = 0.1\mu\text{F}$ the Nyquist plot of the transfer function $G_{\text{BP}}(j\omega)$ is given in Figure 2. Using it, sketch the magnitude Bode plot of the transfer function $G_{\text{BP}}(j\omega)$ in the empty template given in Figure 3.

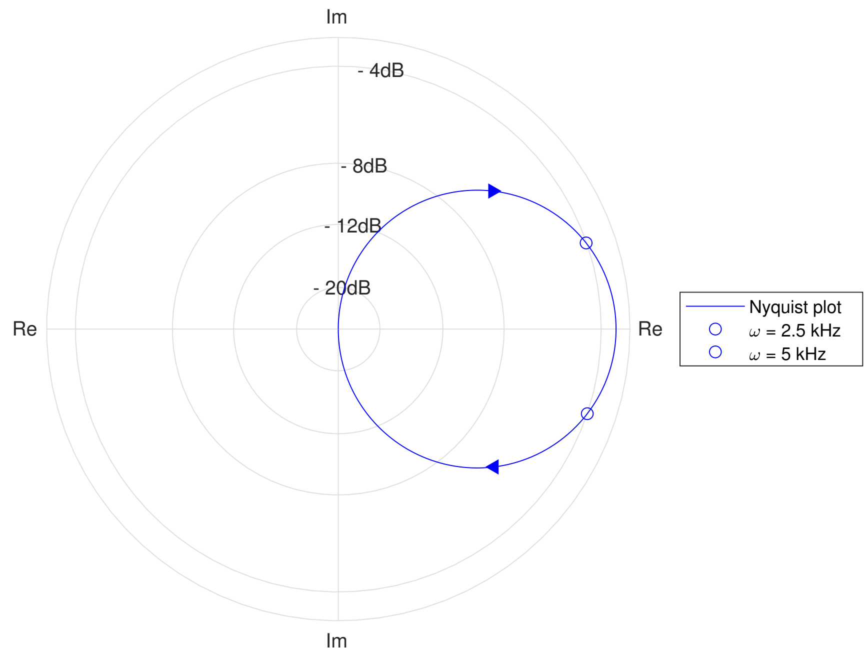

Figure 2: Nyquist plot of $G_{\mathrm{BP}}(j\omega)$.

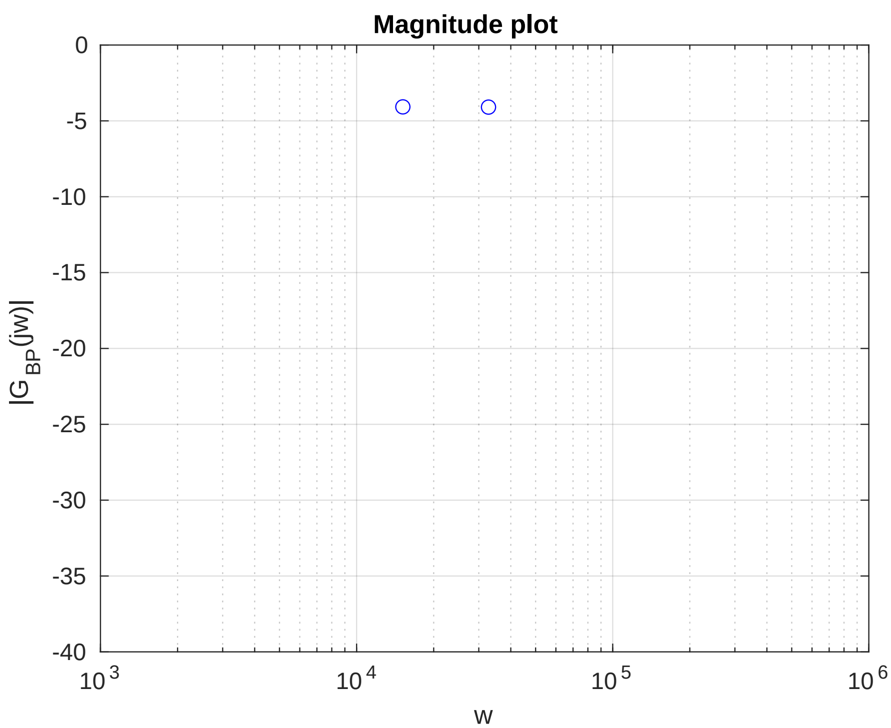

Figure 3: Empty grid for magnitude plot. Annotated points correspond to those in Figure 2.

5. For the values of the circuit parameters given in Part 4, the “pass” frequencies are defined as the ones for which $|G_{\text{BP}}(j\omega)| \geq -4\text{dB}$. Based on the Nyquist plot in Figure 2 estimate the range of “pass” frequencies. Hence, provide an intuitive explanation of why the combined circuit is called a band-pass filter.

Exercise 2

1	2	3	4	5	6	7	Exercise
3	3	3	4	3	4	5	25 Points

Consider the following linear time invariant system with parameters $a, b, c \in \mathbb{R}$:

$$\dot{x}(t) = \overbrace{\begin{bmatrix} -1 & a \\ a & -1 \end{bmatrix}}^{A} x(t) + \overbrace{\begin{bmatrix} 1 \\ b \end{bmatrix}}^{B} u(t)$$
$$y(t) = \underbrace{\begin{bmatrix} c & 0 \end{bmatrix}}_{C} x(t).$$

1. For what values of (a, b, c) is the system controllable?

2. For what values of (a, b, c) is the system observable?

3. For what values of (a, b, c) is the autonomous system (where $u(t) = 0$ for all t) asymptotically stable?

4. For what values of (a, b, c) is the matrix A diagonalizable?

5. Notice that the matrix A is symmetric. For what values of (a, b, c) is it also positive definite?

6. For what values of (a, b, c) with $a \geq 0$ is the system stabilizable?

7. For $(a, b, c) = (2, 1, 1)$, design a feedback controller of the form

$$u(t) = Kx(t) = \begin{bmatrix} k_1 & 0 \end{bmatrix} x(t)$$

such that the closed-loop system $\dot{x}(t) = (A + BK)x(t)$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -3$. Can you do the same for $\lambda_1 = -2$ and $\lambda_2 = -2$?

Exercise 3

1(a)	1(b)	1(c)	1(d)	1(e)	2(a)	2(b)	2(c)	2(d)	Exercise
3	2	2	2	4	2	3	4	3	25 Points

1. Figure 4 shows the Nyquist plot of the transfer function

$$G(s) = \frac{1}{(s+100)(s+2)}. \tag{1}$$

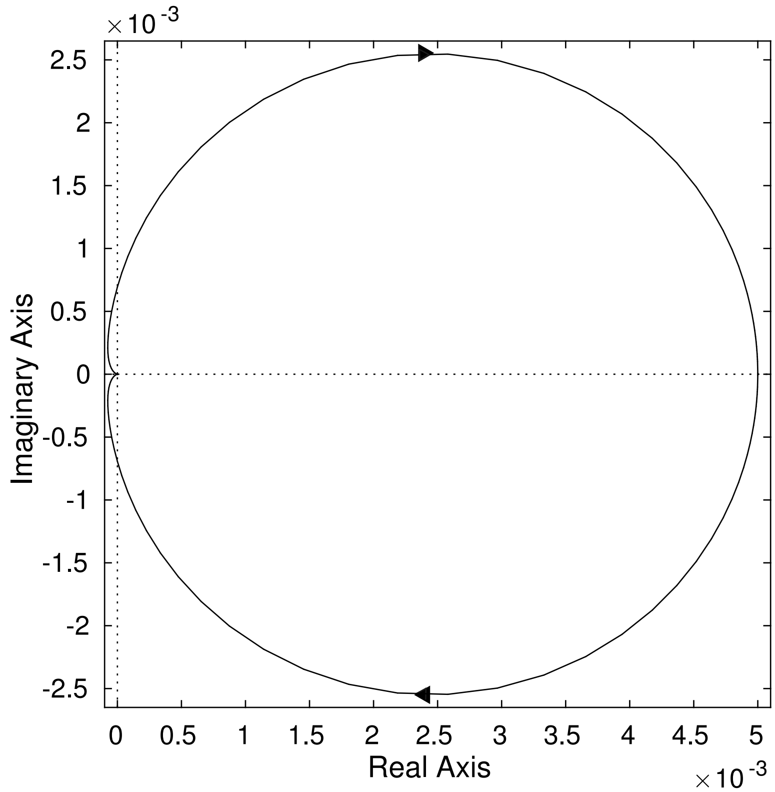

Figure 4: Nyquist plot of $G(s)$.

Use it to match the transfer functions given in (a) - (d) with the Nyquist plots in Figures 5 - 10. Justify your choices with the necessary computations. The same Nyquist plot may be correct for more than one of the systems (a) - (d).

(a) $G_a(s) = \frac{1}{(s+100)(s-2)}$.

(b) $G_b(s) = \frac{2}{(s+100)(s+2)}$.

(c) $G_c(s) = \frac{1}{(s-100)(s-2)}$.

(d) $G_d(s) = \frac{-1}{(s+100)(s+2)}$.

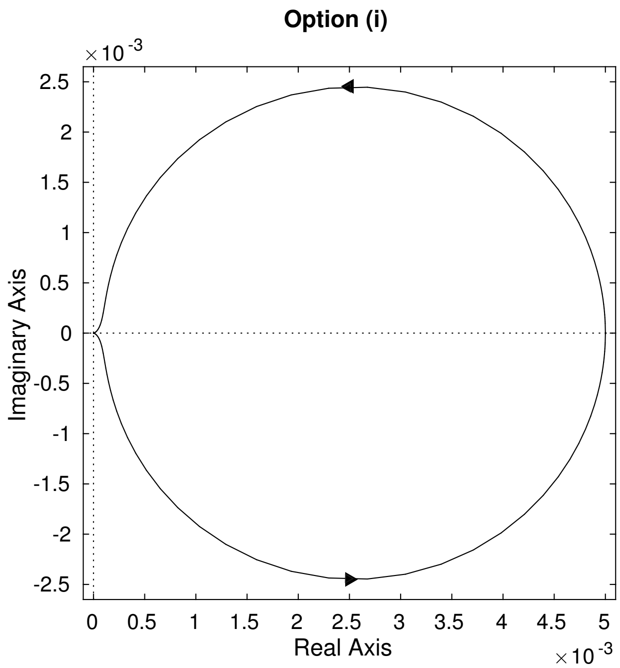

Figure 5: Nyquist plot - option (i)

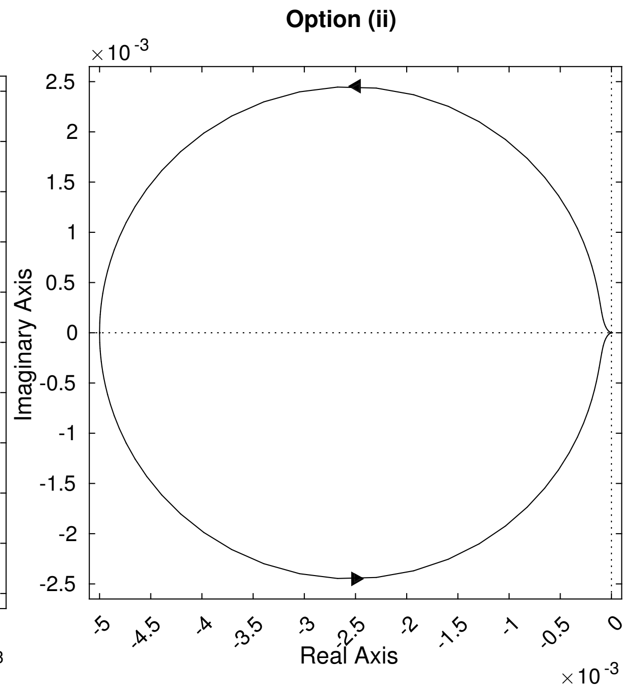

Figure 6: Nyquist plot - option (ii)

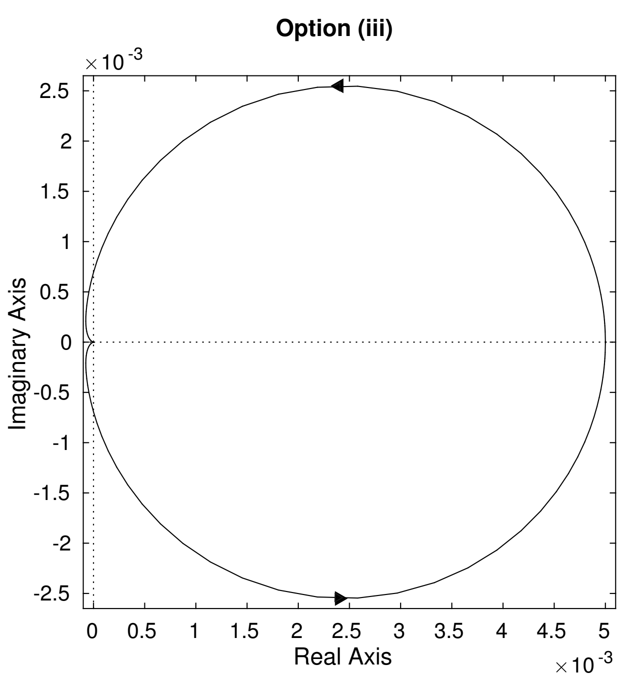

Figure 7: Nyquist plot - option (iii)

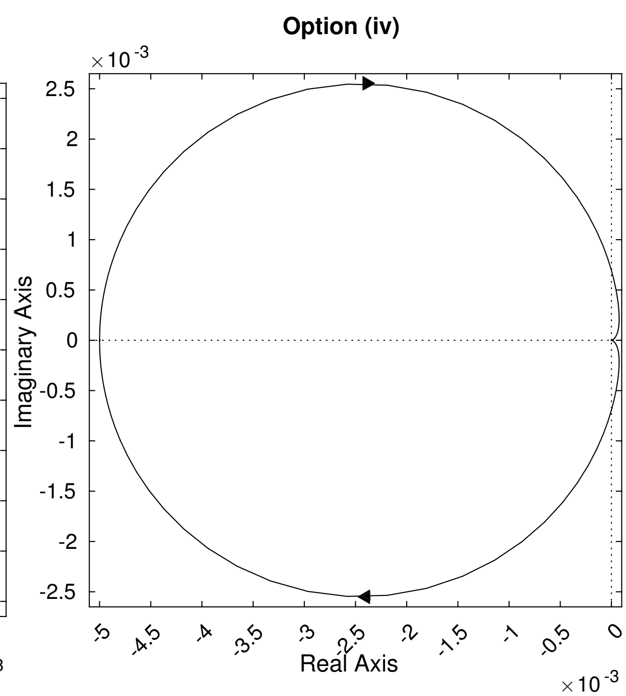

Figure 8: Nyquist plot - option (iv)

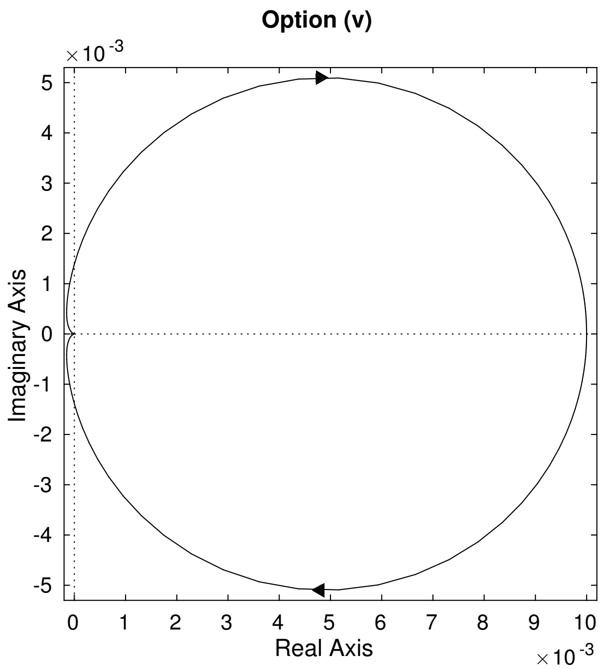

Figure 9: Nyquist plot - option (v)

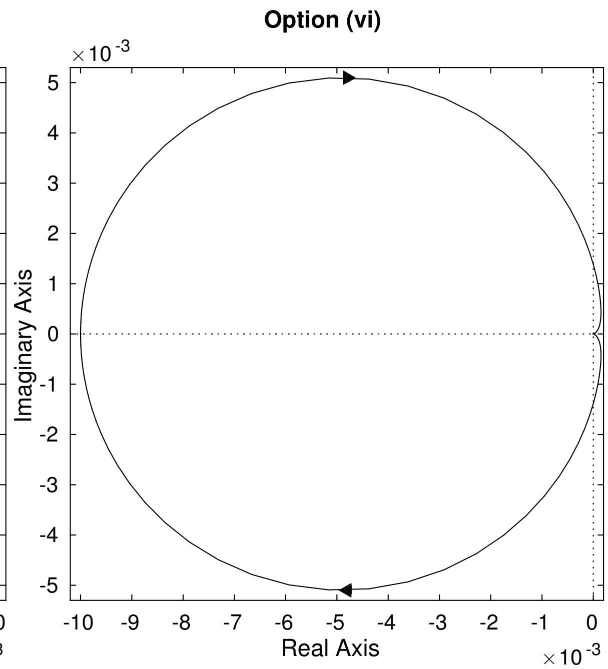

Figure 10: Nyquist plot - option (vi)

(e) Assume that each of the systems (a) - (d) is connected in a feedback loop shown in Figure 11. Comment on the stability of each of the systems under the feedback connection for $k = 400$.

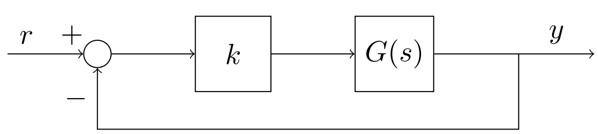

Figure 11: Closed-loop system

2. The transfer function of an inverted pendulum linearised about the upright position is given by

$$G_2(s) = \frac{1}{s^2 - 1}. \tag{2}$$

The input $U(s)$ is the torque applied to the base of the pendulum, whereas the output $Y(s)$ is the angle of the pendulum from the upright position.

(a) Is the system stable?

(b) The system is connected in the feedback loop, shown in Figure 12, with a controller of the form

$$K(s) = K_P + K_D s + \frac{K_I}{s}, \tag{3}$$

where $K_P, K_D, K_I \in \mathbb{R}$ are design parameters. Derive the transfer function $G_{CL}(s)$ of the closed loop system with $R(s)$ as input and $E(s)$ as output. Arrive at:

$$G_{CL}(s) = \frac{E(s)}{R(s)} = \frac{s(s^2-1)}{s^3 + K_D s^2 + (K_P - 1)s + K_I}.$$

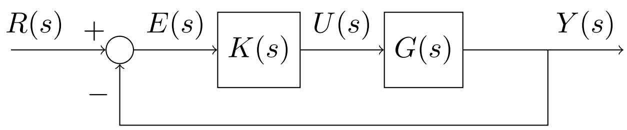

Figure 12: Closed-loop system with a feedback controller

(c) Select K_P, K_D and K_I so that the closed loop system has all poles at $s = -1$.

(d) Assume now that a step input

$$r(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

is applied to the closed loop system. Compute the steady state error

$$\lim_{t \to \infty} e(t)$$

for the parameter values in Part (c).

Exercise 4

1	2	3	4	5	Exercise
3	6	8	6	2	25 Points

For $y(t) \in \mathbb{R}$, $t \geq 0$, $d > 0$ and $k > 0$ a non-linear system is described by the following dynamics

$$\ddot{y}(t) + d\dot{y}(t)^3 + ky(t) = 0\,. \tag{4}$$

1. By selecting appropriate state variables write the system in the following state space form:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -kx_1(t) - dx_2(t)^3\,.\end{aligned}$$

   Hence, show that $(y, \dot{y}) = (0, 0)$ is the only equilibrium of system (4).

2. Can you determine the stability of the equilibrium using linearization?

3. Using the Lyapunov function

$$V(x(t)) = \frac{1}{2}\left(kx_1(t)^2 + x_2(t)^2\right)\,,$$

   show that the equilibrium is stable. Can you also determine whether the equilibrium is asymptotically stable with the same method?

   Hint: You may assume that $S = \mathbb{R}^2$ is an open set.

4. Consider the following set

$$S = \left\{x(t) \in \mathbb{R}^2 \,|\, V(x) \leq K\right\}\,,$$

   for the Lyupanov function in Part 3 and some $K > 0$. Using your answer to Part 3, show that $M = \{(0, 0)\}$ is the largest invariant set contained in the set

$$\bar{S} = \left\{x(t) \in S \,\middle|\, \dot{V}(x(t)) = 0\right\}\,.$$

   Hence, argue that the equilibrium is globally asymptotically stable.

   Hint: You may assume that the set S is compact. Note that the K in the definition of S is arbitrary and that $V(x) \to \infty$ as $\|x\| \to \infty$.

5. Consider now the vector field of the Van der Pol oscillator shown in Figure 13. The oscillator has a unique equilibrium at $\hat{x} = (0, 0)$. Based on the figure and without doing any calculations, comment on the stability of the equilibrium and the general behavior of the system from an arbitrary initial condition $x(0) \in \mathbb{R}^2$.

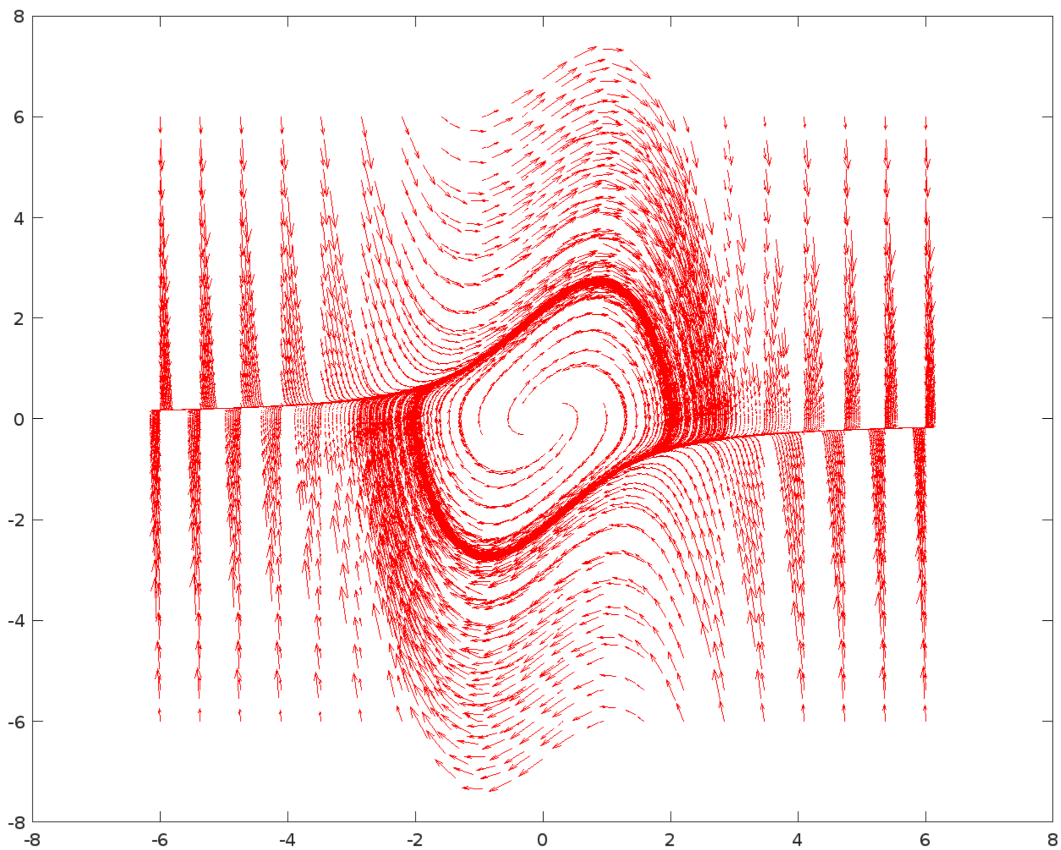

Figure 13: Vector field of the Van der Pol oscillator.