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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4	5	Exercise
3	5	5	4	8	25 Points

A solar panel system can be modelled as the equivalent circuit shown in Figure 1, where the panel outputs a current $i_s(t)$ and a power converter extracts a current $i_p(t)$ from the circuit to transfer the power into a power grid.

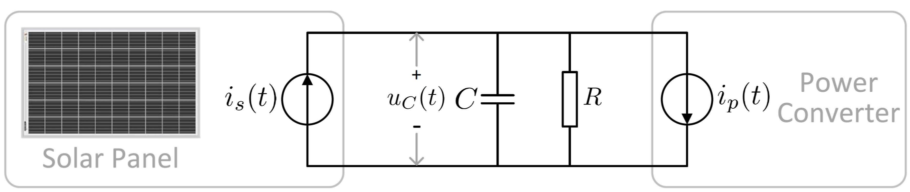

Figure 1: Equivalent circuit of a solar panel.

1. Show that the solar panel can be modelled by a state space system of the form

$$\dot{u}_C(t) = -\frac{u_C(t)}{CR} + \frac{i_s(t)}{C} - \frac{i_p(t)}{C}\,.$$

2. A controller with transfer function $G(s) = K_\text{P} + \frac{K_\text{I}}{s}$ is used to control the value of $i_p(t)$ to regulate the capacitor voltage $u_C(t)$ to a desired reference value $u_\text{ref}(t)$, as shown in Figure 2; K_P and K_I are constant gains. Show that the controller can be modelled in state space form by

$$\dot{z}(t) = u_C(t) - u_\text{ref}(t)$$
$$i_p(t) = K_\text{P}(u_C(t) - u_\text{ref}(t)) + K_\text{I} z(t)$$

where $z(t) \in \mathbb{R}$ (with $z(0) = 0$) plays the role of the controller state, $u_C(t) - u_\text{ref}(t)$ the controller input and $i_p(t)$ the controller output.

Hint: For a signal $h(t)$ $(t \geq 0)$, $\frac{1}{s}L\{h(t)\} = L\{\int_0^t h(\tau)d\tau\}$, where $L\{\cdot\}$ denotes the Laplace transform.

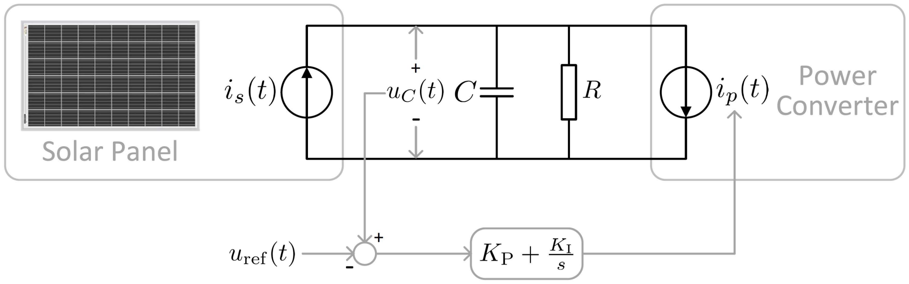

Figure 2: Equivalent circuit of a solar panel with a PI controller.

3. Derive the matrices A, B, C, and D for the state space model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,, \\ y(t) &= Cx(t) + Du(t)\,, \end{aligned} \tag{1}$$

of the closed-loop system in Figure 2, where $x(t) = \begin{bmatrix} u_C(t) \\ z(t) \end{bmatrix}$, $u(t) = \begin{bmatrix} u_{\mathrm{ref}}(t) \\ i_s(t) \end{bmatrix}$, and $y(t) = u_C(t)$.

4. Assuming that both the reference voltage $u_{\mathrm{ref}}(t) = u_0$ and the solar panel current $i_s(t) = i_0$ are constant, determine the equilibrium $\hat{x} = \begin{bmatrix} \hat{u}_C \\ \hat{z} \end{bmatrix}$ of system (1).

5. Assuming $C > 0$ and $R > 0$, derive conditions for K_{P} and K_{I} such that the equilibrium in part 4 is asymptotically stable.

Exercise 2

1	2	3	4	5	Exercise
4	4	4	7	6	25 Points

Consider the block diagram in Figure 3.

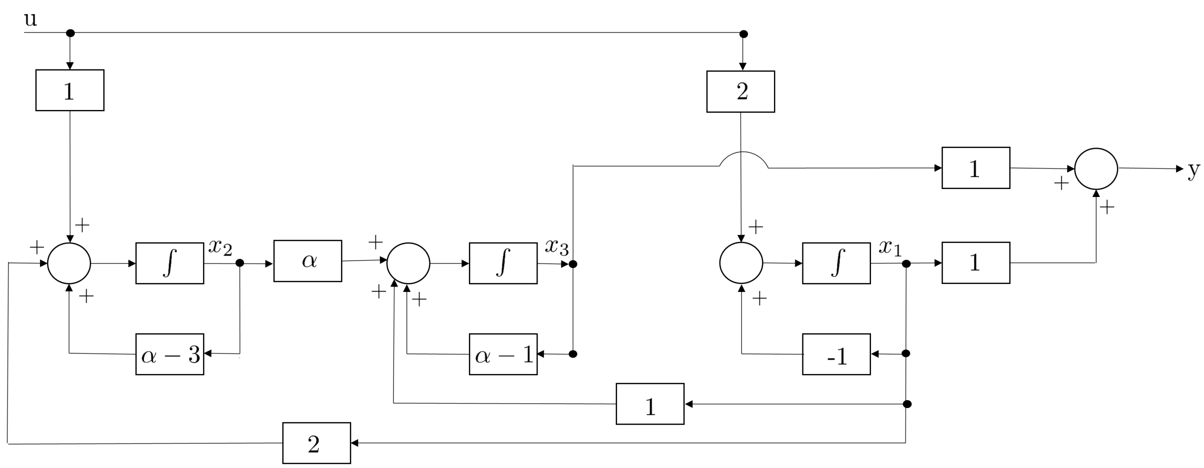

Figure 3: Representation of a parametric block diagram.

1. Using $x(t) = \big(x_1(t),\quad x_2(t),\quad x_3(t)\big) \in \mathbb{R}^3$, derive the state space matrices A, B, C, and D for the state space model represented by the block diagram in Figure 3:

$$\dot{x}(t) = Ax(t) + Bu(t)\,,$$
$$y(t) = Cx(t) + Du(t)\,,$$

Hint: The input to an integrator is the derivative of its output.

2. For which values of α is the system asympotically stable?

For the rest of the exercises, consider the following system:

$$\dot{x} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} \gamma \\ 0 \\ \beta \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \delta \end{bmatrix} x$$

3. For which value of δ is the system observable?

4. Because actuators are expensive you can only afford to inject the input in the dynamics of one of the states, either x_1 (corresponding to $\gamma = 1,\ \beta = 0$), or x_3 ($\gamma = 0, \beta = 1$). Compute the reachable space for the two choices. Hence argue which is the better choice.

5. Due to miscommunication, the procurement department of your employer bought an actuator for x_1 so now you have to work with $\gamma = 1$ and $\beta = 0$. Is it possible to select a feedback gain $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$ such that the closed loop system with state feedback $u(t) = Kx(t)$ has poles at -5, -1 and 0? Will the closed loop system be stable?

Exercise 3

1(a)	1(b)	1(c)	1(d)	2(a)	2(b)	2(c)	Exercise
8	3	4	4	2	2	2	25 Points

1. Consider the following transfer functions of observable and controllable systems.

   i $G_1(s) = \frac{-10^3}{(s+10)^2(s^2+s+1)}$

   ii $G_2(s) = \frac{-10^3}{(s+10)(s-10)(s^2+s+1)}$

   iii $G_3(s) = \frac{-10^3}{s^2(s+10)(s^2+s+1)}$

   iv $G_4(s) = \frac{-10^3}{s(s+10)(s^2+s+1)}$

   (a) Compute the poles of all four transfer functions. Hence, whenever possible, determine whether the corresponding system is asymptotically stable, marginally stable or unstable. Justify your answers.

   (b) The Nyquist plot of $G_1(s)$ is shown in Figure 4.

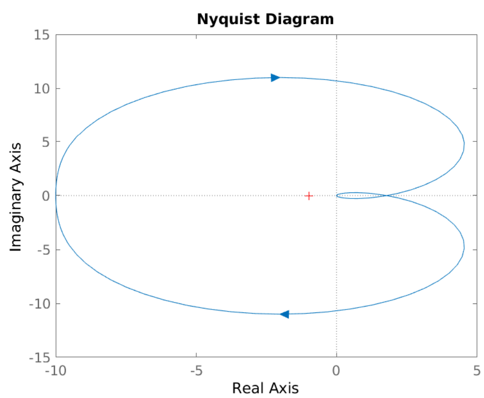

Figure 4: Nyquist plot of $G_1(s)$.

   $G_1(s)$ is inserted in a negative feedback loop with gain K, leading to the closed-loop transfer function $H(s) = \frac{KG_1(s)}{1+KG_1(s)}$. For which of the following values of K is the closed-loop stable? For $K = \frac{1}{5}$ or for $K = -\frac{1}{5}$? Justify your answer.

   (c) For $G_1(s)$ and $G_2(s)$ compute, whenever possible, the steady-state response to a sinusoidal input with amplitude 2 and frequency 10 [rad/sec].
   Hint: You may assume that $\sqrt{99^2+10^2} \approx 99$ and that $\tan^{-1}(99/10) \approx 84°$ and $\tan^{-1}(10/-99) \approx -5.77°$. In addition, recall the following

$$\angle a + ib = \begin{cases} -\pi + \tan^{-1}\left(\frac{b}{a}\right) & \text{if } a < 0 \text{ and } b < 0\,, \\ \tan^{-1}\left(\frac{b}{a}\right) & \text{if } a > 0\,, \\ \pi + \tan^{-1}\left(\frac{b}{a}\right) & \text{if } a < 0 \text{ and } b \geq 0\,. \end{cases}$$

   (d) Which of the Bode plots in Figure 5 corresponds to the transfer function $G_2(s)$? Is there resonance? Justify your answers.

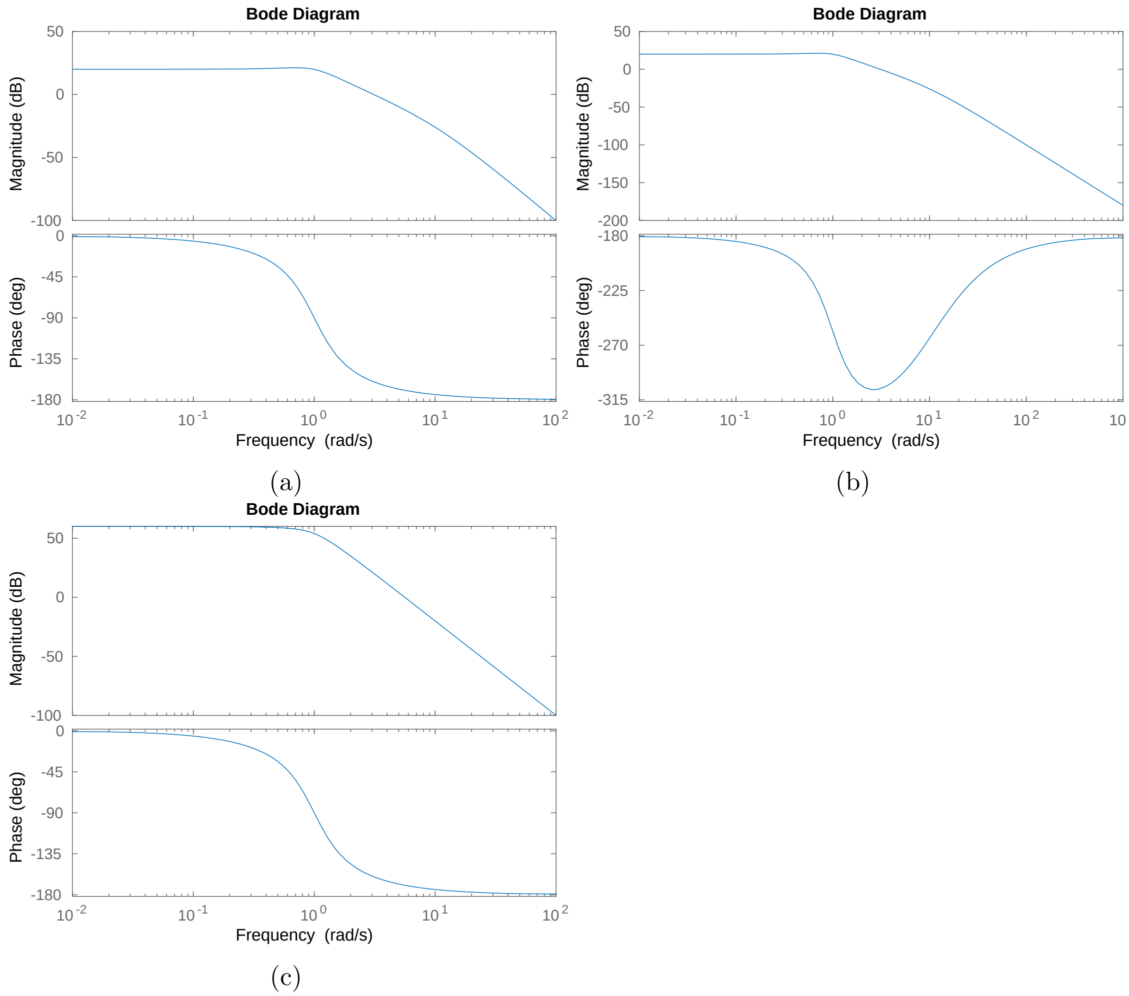

Figure 5: Candidate Bode plots for $G_2(s)$.

2. Consider the following system in state-space form

$$\dot{x}(t) = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)\,, \tag{2}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)\,, \tag{3}$$

with $x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$.

(a) Show that the transfer function of the system is $G(s) = \frac{1}{s+2}$.

(b) Can we infer whether the system is asymptotically stable, marginally stable or unstable from the transfer function? Justify your answer.

(c) What can you conclude about the observability and controllability of the system without computing the observability and controllability matrices?

Exercise 4

1	2	3	4	Exercise
3	8	9	5	25 Points

For $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$, $t \geq 0$, a non-linear system is described by the following dynamics

$$\begin{aligned} \frac{d}{dt}x_1(t) &= -4x_1(t) + 4x_1(t)^3 - \frac{x_2(t)}{1 + x_2(t)^2}, \\ \frac{d}{dt}x_2(t) &= \frac{x_1(t)}{1 + x_2(t)^2}. \end{aligned} \tag{4}$$

1. Show that $\hat{x} = (0, 0)$ is the only equilibrium of system (4).

2. Using Lyapunov's direct method with the Lyapunov function $V(x) = x_1^2 + x_2^2$, determine whether the equilibrium $\hat{x}$ is stable. Can you also determine whether it is asymptotically stable using the same method and Lyapunov function?

   Hint: You may assume that the set $S = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$ is open.

3. Using the same Lyapunov function, argue that the set

   $$\Omega := \{(x(t)) \in \mathbb{R}^2 \mid x_1(t)^2 + x_2(t)^2 \leq 1\}$$

   is invariant. Using LaSalle's theorem show that all trajectories starting in Ω converge to $\hat{x}$. Show that Ω is the largest region of attraction that can be estimated using level sets of $V(x)$.

   Hint: You may assume Ω is compact. The level sets of $V(x)$ are $\mathcal{L}(l) = \{x | V(x) \leq l\}$.

4. Figure 6 shows a phase plane plot of the system (4) with its vector field and trajectories for various values of the state. Argue that Ω is not the maximal region of attraction.

   Hint: Identify initial states indicated in Figure 6 that are in the region of attraction, but not in Ω.

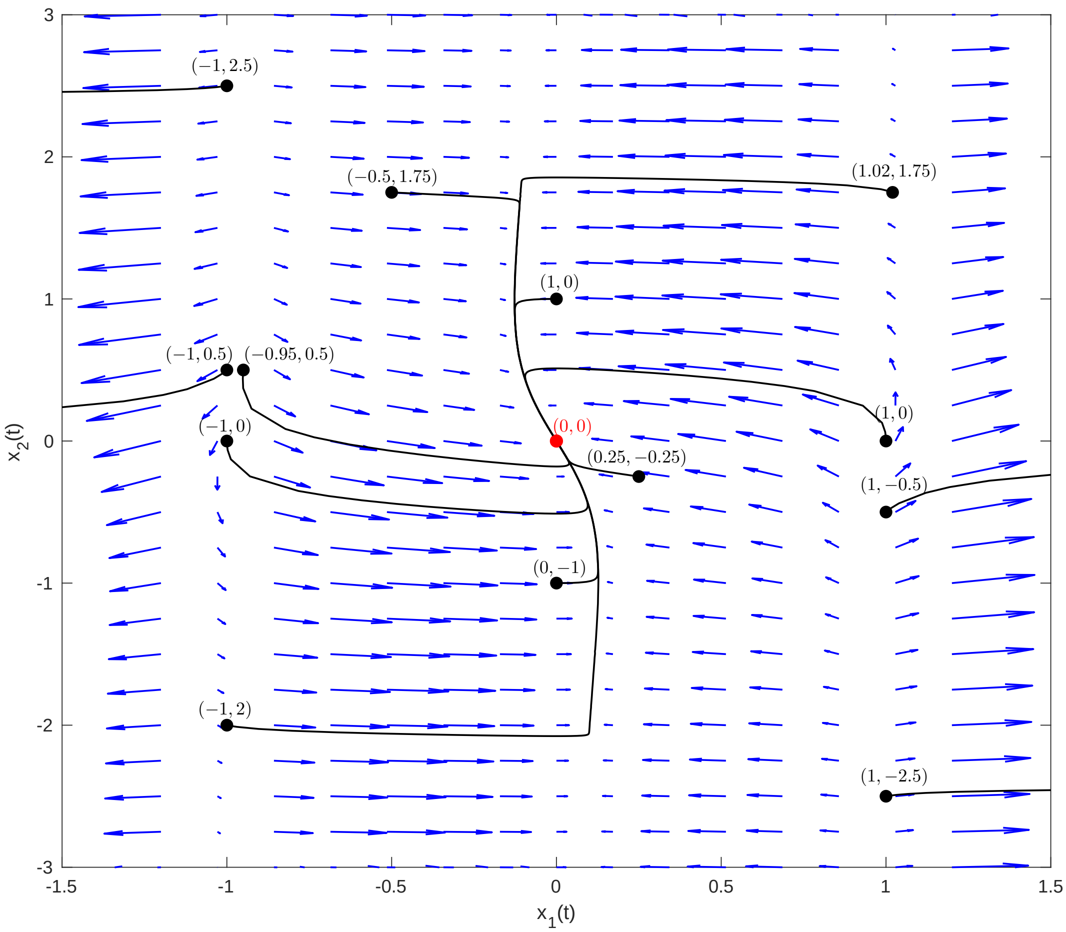

Figure 6: The phase portrait of the system (4) with vector field, and trajectories for various values of the initial state.