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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4	5	Exercise
5	5	4	5	6	25 Points

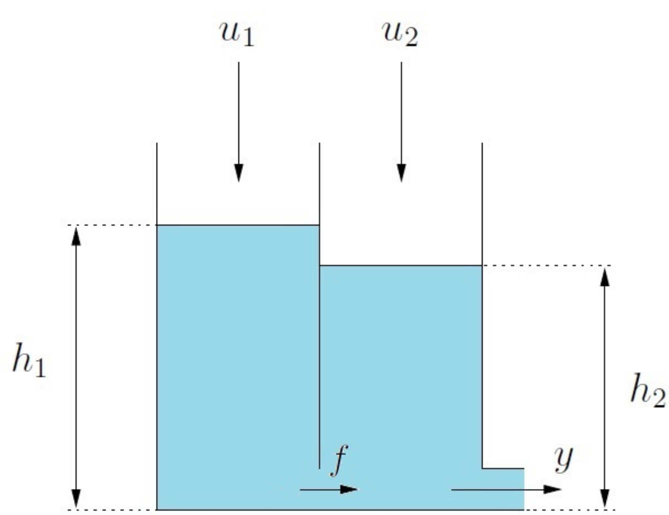


Figure 1: A water flow system with two coupled tanks.

Consider the water flow system of two coupled tanks as depicted in Figure 1. The flows of water into the left and right tanks are denoted by u_1 and u_2, respectively (both in cubic meters m^3). The water levels of the left and right tanks are denoted by h_1 and h_2, respectively (both in m). The flow y (in m^3) out of the right tank is assumed to be proportional to the water level in this tank, that is,

$$y(t) = \alpha h_2(t),$$

with $\alpha > 0$ (in m^2). Moreover, the flow f (in m^3) between the left and right tanks is assumed to be proportional to the difference between the water levels, that is,

$$f(t) = \beta(h_1(t) - h_2(t)),$$

with a flow from left to right defined as positive, $\beta > 0$ (in m^2). For the sake of simplicity, we ignore the physical constraint that u_1, u_2, h_1, h_2, and y have to remain positive; in practice this can be done by re-defining them as deviations from some nominal values. Assume that the area of the bases of the tanks are v_1 and v_2, respectively (both in m^2).

1. Determine the matrices A, B and C so that the dynamics of the considered water flow system become

$$\dot{h}(t) = A \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} + B \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad \text{and} \quad y(t) = C \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}.$$

2. Assume α, β, v_1, and v_2 are selected so that $A = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Is the system stable? Is it controllable? Is it observable?

3. For the values given in Part 2, determine the transfer function, $G(s)$, of the system.

4. Assume that the input $u_1(t) = u_2(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$ is applied to the system. For the values given in Part 2, compute $\lim_{t\to\infty} y(t)$.
   Hint: Recall the final value theorem.

5. More generally, assume that the input $u(t) = \begin{cases} \overline{u} & t \geq 0 \\ 0 & t < 0 \end{cases}$ is applied to the system, for some $\overline{u} = \begin{bmatrix} \overline{u}_1 \\ \overline{u}_2 \end{bmatrix} \in \mathbb{R}^2$. For the values given in Part 2, derive an expression for $|\lim_{t\to\infty} y(t)|$ as a function of $\overline{u}$. If $\|\overline{u}\| = \sqrt{\overline{u}_1^2 + \overline{u}_2^2} = 1$, determine values of $\overline{u}$ that maximise and minimise $|\lim_{t\to\infty} y(t)|$ and the corresponding maximum and minimum.

Exercise 2

1	2	3	4	5	Exercise
3	6	3	7	6	25 Points

Consider the system

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -2 & -1 & a \\ 0 & a & 1 \\ 0 & 0 & -3 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}}_{C} x(t)$$

where $t \in \mathbb{R}_+$, $x(t) \in \mathbb{R}^3$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$. The constant $a \in \mathbb{R}$ is a parameter of the system.

1. For which values of a is the system asymptotically stable?

2. Your friend from EPFL claims that for all values of a, the system can be driven to the state
$$x(T) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$
for some time $T > 0$, starting at $x(0) = 0$, by selecting an appropriate input. Do you agree with him? If not, indicate for which values of a the given $x(T)$ is reachable.

3. For which values of a is the system observable?

4. How many poles will the transfer function of the system have?
Hint: Recall the tests rank $\begin{bmatrix} C \\ \lambda I - A \end{bmatrix}$ and rank$[B \ \lambda I - A]$. The answer may depend on the value of a.

5. Assume that for some value of a you are to solve the equation $PA + A^T P = -I$, where I is the identity matrix, to obtain a symmetric positive definite matrix $P \in \mathbb{R}^{3\times 3}$. What does this tell you about the stability of the system?
Show that in this case there exists an invertible matrix $T \in \mathbb{R}^{3\times 3}$ and a symmetric, positive definite matrix $R \in \mathbb{R}^{3\times 3}$, such that the matrix $\hat{A} = TAT^{-1}$ satisfies
$$\hat{A} + \hat{A}^T = -R.$$
Hint: Every real symmetric positive definite matrix, P, has a real symmetric positive definite square root, Q, such that $P = Q^2$. The expressions for T and R may be given in terms of such a Q, without computing P or Q explicitly.

Exercise 3

1	2	3	4	5	Exercise
4	5	8	5	3	25 Points

Consider the linear system

$$\Sigma_0 : \quad \begin{aligned} \dot{x}_1(t) &= A\,x_1(t) + B\,u_1(t) \\ y_1(t) &= C\,x_1(t) + D\,u_1(t) \end{aligned}$$

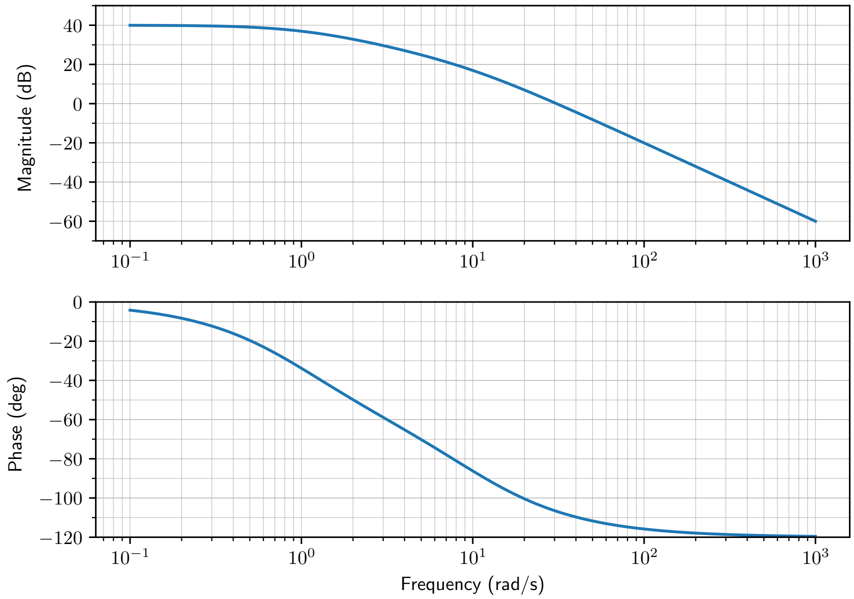
given by the matrices

$$A = \begin{bmatrix} -12 & -11 \\ 2 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 500 \end{bmatrix}, \ D = 0. \tag{1}$$

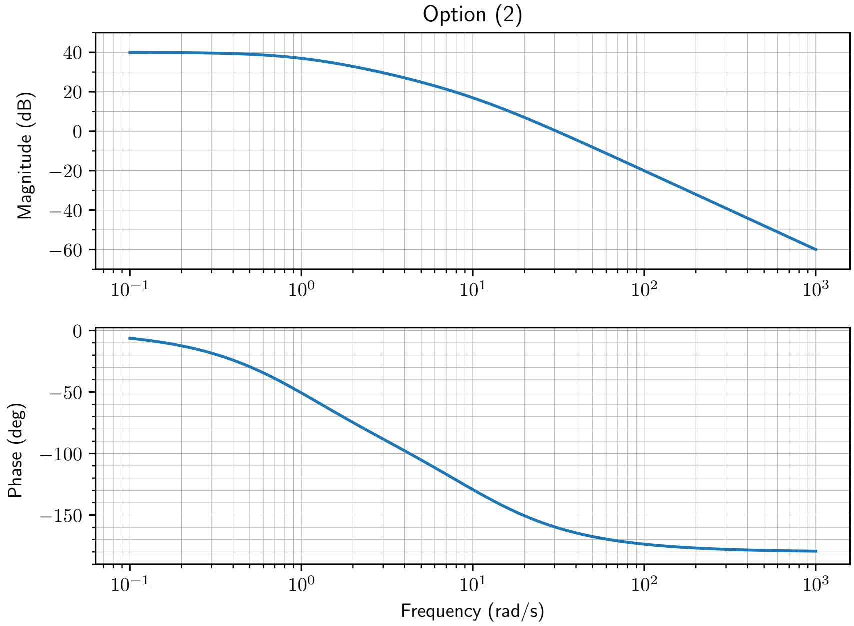
1. Compute the transfer function $G_0(s)$ of the system Σ_0.

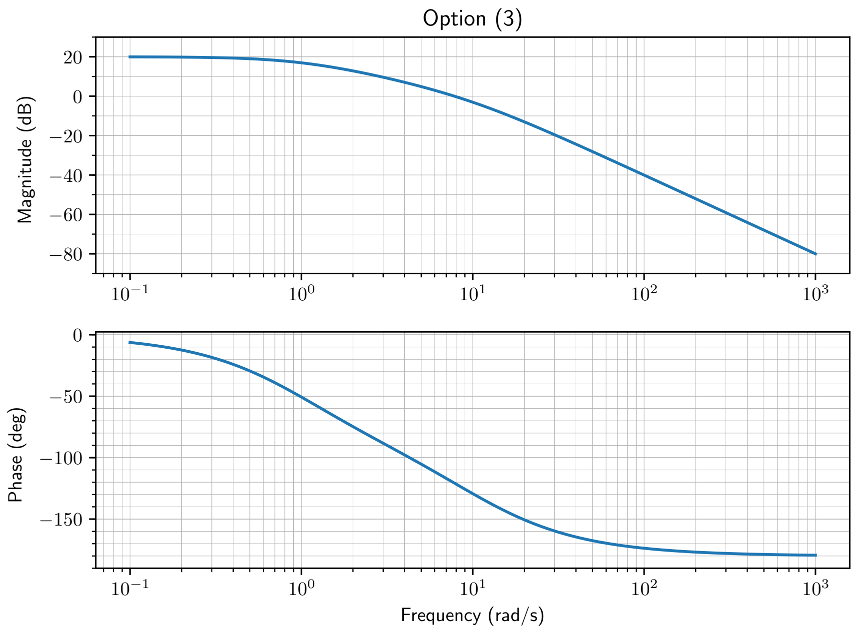
2. Pick the Bode plot corresponding to the transfer function $G_0(s)$ from the 4 options given below. Justify your choice with appropriate reasoning.
Hint: Pay close attention to the values on the magnitude and phase axes in the given plots.

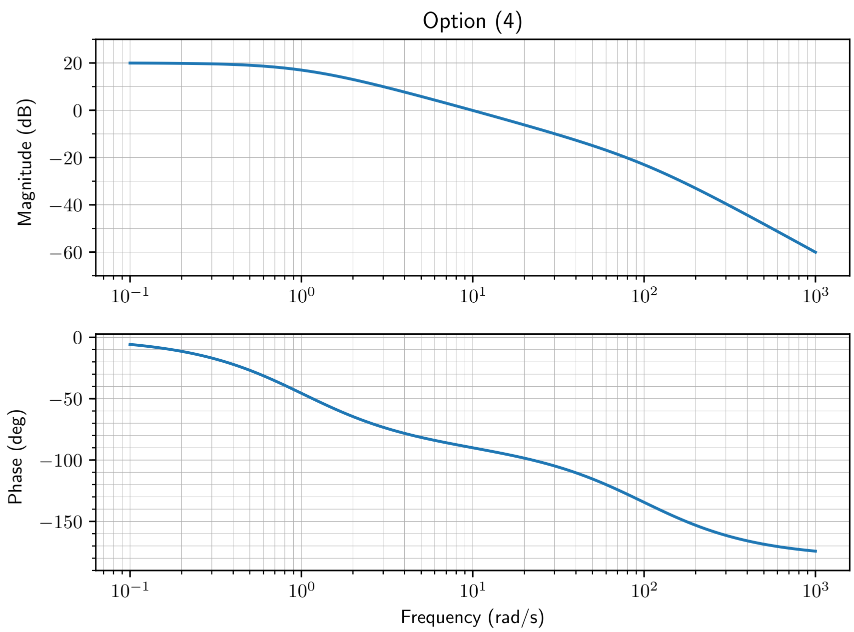
Option (1)



Option (2)




Option (3)
Magnitude (dB)
20
0
−20
−40
−60
−80
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Phase (deg)
0
−50
−100
−150
Frequency (rad/s)


Option (4)
Magnitude (dB)
20
0
−20
−40
−60
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Phase (deg)
0
−50
−100
−150
Frequency (rad/s)

Next, consider a single input single output system Σ_1 connected to a controller Σ_2 as shown in Figure 2. Consider the controller Σ_2 given by $y_2(t) = Ke(t)$ for some $K > 0$ and assume that the transfer function of Σ_1 only has poles with negative real part.

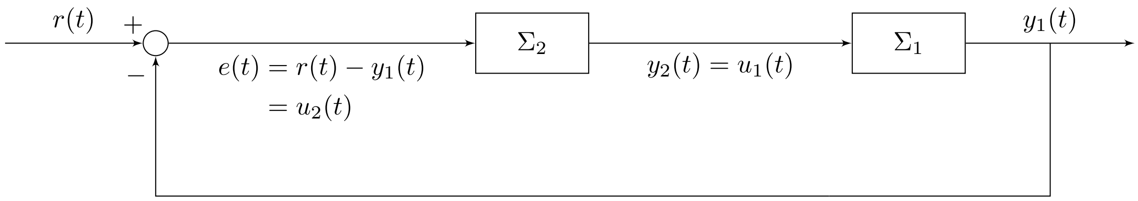


Figure 2: Feedback system

3. The Bode plot of system Σ_1 is given in Figure 3.
    - Is it possible to use the Bode stability criterion to determine the stability of the closed loop system in Figure 2?

   If so, use the given Bode plot to estimate
    - the phase margin,
    - gain margin
    - and the range of positive gains, K, for which the closed loop system in Figure 2 is asymptotically stable.

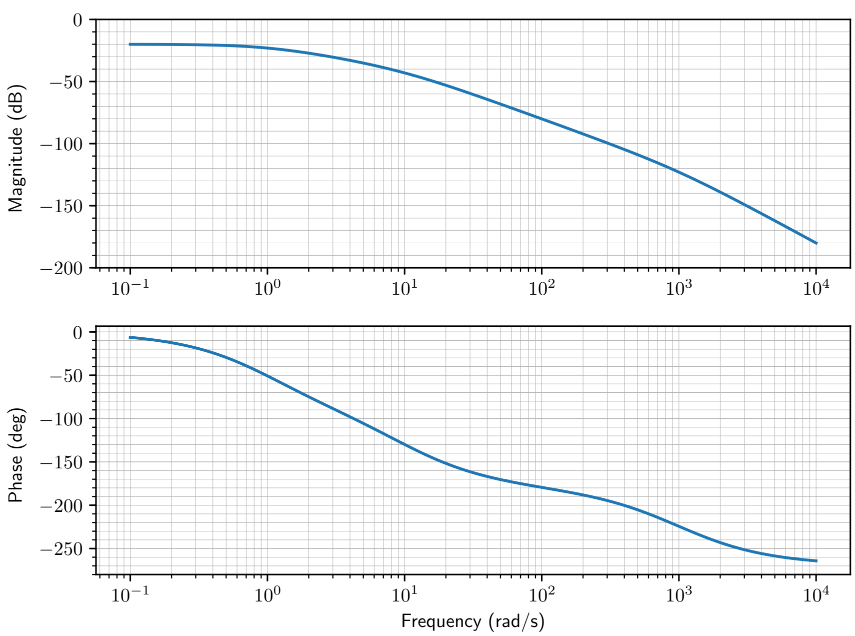


Figure 3: Bode plot for Σ_1

4. The Nyquist plot of Σ_1 is given in Figure 4 with a zoomed-in plot of the area near the origin in Figure 5. Estimate again the range of positive K for which the closed loop system shown in Figure 2 is asymptotically stable using the Nyquist stability criterion.

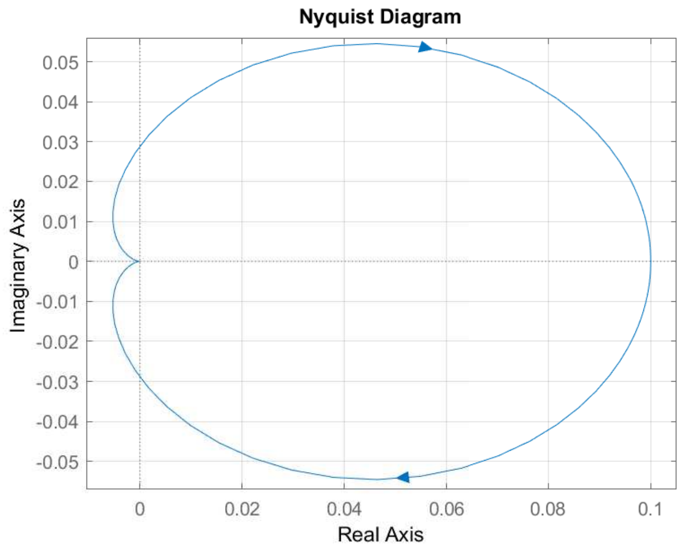


Figure 4: Nyquist plot of Σ_1

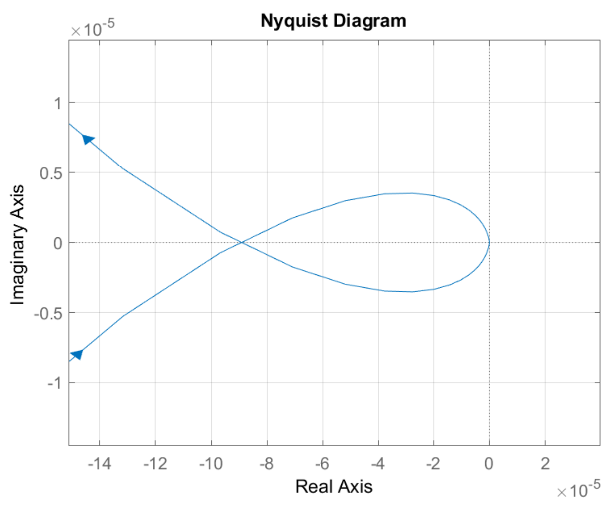


Figure 5: Zoomed in Nyquist plot of Σ_1

5. Consider now the gain $K = 2 * 10^4$. How many poles with positive real part do you expect the closed loop system to have?

Exercise 4

1	2	3	4	5	Exercise
3	4	8	7	3	25 Points

The spread of an infectious disease through a population can be described by a compartmental model of the form

$$\begin{aligned} \dot{x}_1(t) &= \alpha - \mu x_1(t) - \beta x_1(t) x_2(t) \\ \dot{x}_2(t) &= \beta x_1(t) x_2(t) - (\gamma + \mu) x_2(t), \end{aligned} \tag{2}$$

where x_1 is the amount of susceptible individuals in the population (those who can potentially be infected) and x_2 is the amount of infected individuals. The parameters $\beta > 0$ and $\gamma > 0$ represent the infection and recovery rates, whereas the parameters $\alpha > 0$ and $\mu > 0$ represent the birth and death rates. We define the basic reproduction rate by

$$R_0 = \frac{\alpha\beta}{\mu(\gamma + \mu)}$$

and, to keep the model realistic, assume that initially the amounts of both susceptible and infected individuals are non-negative, i.e.

$$x(0) \in S_0 = \{x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}.$$

1. Argue that the positive orthant, S_0, is an invariant set for this system.
2. Show that the system has two equilibria, one at $x = (\frac{\alpha}{\mu}, 0)$, where the population is "disease free", and the other at $x = (\frac{\gamma+\mu}{\beta}, \frac{\mu}{\beta}(R_0 - 1))$, where the disease is "endemic".
3. Analyse the stability of both equilibria using linearization for $R_0 < 1$, $R_0 = 1$ and $R_0 > 1$. Give an interpretation of your results: how does the basic reproduction number affect the spread of the virus?
   Hint: The roots of a second order polynomial $\lambda^2 + a\lambda + b = 0$ have negative real parts if and only if $a > 0$ and $b > 0$. Not all equilibria are relevant for all values of R_0.
4. Note that in the case $R_0 = 1$ the two equilibria coincide; let $x^* = (\frac{\alpha}{\mu}, 0) = \frac{\gamma+\mu}{\beta}, \frac{\mu}{\beta}(R_0 - 1))$ denote this unique equilibrium. Consider the function

   $$V(x) = x_1^* \left(\frac{x_1}{x_1^*} - \ln \frac{x_1}{x_1^*} - 1 \right) + x_2 - x_2^*,$$

   and the set

   $$S_K = \{x \in S_0 \mid V(x) \leq K,\ K > 0\}.$$

   (a) Compute $\frac{d}{dt} V(x(t))$ and show that S_K is an invariant set for system (2).
   Hint: Note that $S_K \subseteq S_0$.

   (b) Use LaSalle's theorem to show that all trajectories starting in S_K converge to x^* as $t \to \infty$.
   Hint: You may assume that the set S_K is compact for all $K > 0$.

5. One strategy to combat infection is to vaccinate the population at birth. With a vaccination rate of $p \in [0, 1]$ the dynamics then become

$$\begin{aligned} \dot{x}_1(t) &= (1-p)\alpha - \mu x_1(t) - \beta x_1(t) x_2(t) \\ \dot{x}_2(t) &= \beta x_1(t) x_2(t) - (\gamma + \mu) x_2(t). \end{aligned} \tag{3}$$

   Scientists have estimated the basic reproduction rate of COVID-19 to be $R_0 \approx 3$. Which is the minimum vaccination rate p that the government has to adopt to eradicate COVID-19 from the population?