Automatic Control Laboratory
ETH Zürich
Prof. Florian Dörfler

D-ITET
Summer 2019
15.08.2019

Signals and Systems II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	Exercise
10	11	6	27 Points

Represented in Figure 1 is a simplified DC/DC *Buck* converter. The scope of the device is to modulate the ratio $V(t)/V_{in}$ between the output voltage $V(t)$, which supplies a load I_{load}, and the input voltage V_{in}. The desired ratio can be obtained through a proper choice of the behaviour of the switch S that, at each time instant, can be either open or closed.

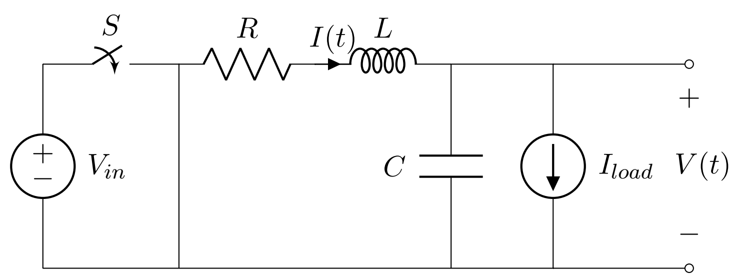

Figure 1: Buck converter electrical model

Notation. R, L, C, V_{in}, I_{load}, as reported in Figure 1, are constant and positive electrical parameters representing the resistance, inductance, capacitance, input voltage and load current of the device, respectively.

1. Let the state be represented by $x(t) = \begin{bmatrix} V(t) & I(t) \end{bmatrix}^T$. Moreover, call f_c and f_o the functions that describe the dynamics of the circuit when the switch is closed and open, respectively.

   (a) Write down the state-space model for the two possible configurations of the device, $\dot{x}(t) = f_c(x(t))$ and $\dot{x}(t) = f_o(x(t))$.

   A typical operation that is performed on switched electrical models is called *state-space averaging*, which consists of computing a unique average dynamics, weighted over the average time spent in each configuration:

$$\begin{aligned}\dot{x}(t) &= u f_c(x(t)) + (1-u) f_o(x(t)) \\ &= f_{avg}(x(t)).\end{aligned}$$

The new parameter $u \in [0,1]$ is the average fraction of time the system works in the "closed" configuration. As a consequence, $1-u$ represents the fraction of time when the switch is open.

(b) Compute the average state-space model of the device, $\dot{x}(t) = f_{avg}(x(t))$.

2. Consider, from now on, $u = u(t)$ as the control input for the average system and, moreover, let $I_{load} = 0$.

   (a) Bring the equations in the form $\dot{x}(t) = Ax(t) + Bu(t)$, and check the stability of the system.

   (b) Compute the equilibria of the system as a function of the constant input $\bar{u} \in [0,1]$. Then, verify that the main relationship of the device,

   $$V/V_{in} \leq 1,$$

   holds at the equilibria.

   (c) Find the expression for the admissible equilibrium set $\mathcal{S}$, i.e., the set of equilibria compatible with an admissible constant input $\bar{u}$:

   $$\mathcal{S} = \left\{ x \in \mathbb{R}^2 \mid Ax + B\bar{u} = 0,\ \bar{u} \in [0,1] \right\},$$

   and represent it graphically in the V-I plane.

3. Answer the following questions and motivate.

   (a) Is it possible to stabilize the system at $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ with a constant input $\bar{u} \in [0,1]$?

   (b) Would your answer to question 3(a) change if $\bar{u}$ was not constrained to $\bar{u} \in [0,1]$, but instead $\bar{u} \in \mathbb{R}$?

   (c) Are your answers to questions 3(a) and 3(b) in contrast with the outcome of the stability test done in question 2?

Exercise 2

1	2	3	4	Exercise
4	5	6	8	23 Points

Consider the system S1 with the following state and output equations:

$$
\begin{aligned}
\dot{x}_1(t) &= \underbrace{\begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}}_{\triangleq A_1} x_1(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\triangleq B_1} u_1(t), \\
y_1(t) &= \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{\triangleq C_1} x_1(t),
\end{aligned} \qquad \text{(S1)}
$$

where $x_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$.

1. (a) Is the system S1 stable?
   (b) Is the system S1 controllable?
   (c) Is the system S1 observable?

2. Consider another system S2. Denote by $x_2 \in \mathbb{R}$ the state of S2, $u_2 \in \mathbb{R}$ the input of S2 and $y \in \mathbb{R}$ the output of S2. The block diagram of S2 is given in Figure 2, wherein $U_2(s)$, $X_2(s)$ and $Y(s)$ are the Laplace transforms of $u_2(t)$, $x_2(t)$ and $y(t)$ respectively.

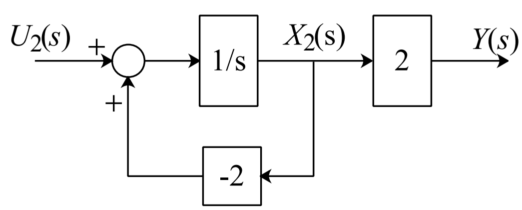

Figure 2: Block diagram of S2.

   (a) Find the transfer functions $\frac{X_2(s)}{U_2(s)}$ and $\frac{Y(s)}{U_2(s)}$.
   (b) Give the state-space equations of S2 in the form $\dot{x}_2(t) = A_2 x_2(t) + B_2 u_2(t)$ and $y(t) = C_2 x_2(t) + D_2 u_2(t)$ where $A_2, B_2, C_2, D_2 \in \mathbb{R}$. Assume that all initial conditions are 0.

3. Now interconnect S2 to S1 to form a cascade system S as shown in Figure 3. Let $x = \begin{bmatrix} x_{11} & x_{12} & x_2 \end{bmatrix}^T$ be the state of S.

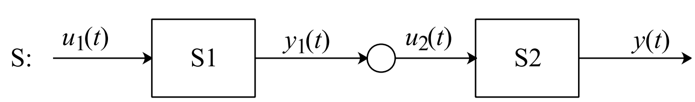

Figure 3: Control system diagram of S.

Suppose that the system S has the following state-space equations:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t),\end{aligned} \tag{S}$$

with $u(t) = u_1(t)$. Determine the matrices A, B, C and D.

4. (a) Show that the system S is unstable.

   (b) Suppose the initial state of S is $x(0) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Derive the relationship of a, b and c to make the zero input transition of S finite as $t \to \infty$.
   Hint: The transition matrix $\phi(t) = e^{At}$ of S is

$$\begin{bmatrix} \frac{4}{5}e^{t} + \frac{1}{5}e^{-4t} & \frac{1}{5}e^{t} - \frac{1}{5}e^{-4t} & 0 \\ \frac{4}{5}e^{t} - \frac{4}{5}e^{-4t} & \frac{1}{5}e^{t} + \frac{4}{5}e^{-4t} & 0 \\ \frac{8}{15}e^{t} + \frac{3}{10}e^{-4t} - \frac{5}{6}e^{-2t} & \frac{2}{15}e^{t} - \frac{3}{10}e^{-4t} + \frac{1}{6}e^{-2t} & e^{-2t} \end{bmatrix}$$

   (c) Your friend from EPFL believes that your answers to 4(a) and 4(b) contradict each other. Explain to your friend why this is not the case.

Exercise 3

1	2	3	Exercise
8	7	10	25 Points

We have seen the Lyapunov methods to analyze the stability of nonlinear **continuous-time** systems. These methods also have a version for nonlinear **discrete-time** systems such as:

$$x(k+1) = f(x(k))$$

Lyapunov first method for discrete-time:
The equilibrium $\tilde{x}$ is

i. **Locally asymptotically stable** if the linearized system has all eigenvalues with norm strictly less than 1.

ii. **Unstable** if the linearized system has at least one eigenvalue with norm strictly larger than 1.

(Remember that for **discrete-time** systems we look at the norm of the eigenvalues, instead of looking at the real part as for **continuous-time** systems)

Lyapunov second method for discrete-time:
If there is an open set $S \subseteq \mathbb{R}^n$ with equilibrium $\tilde{x} \in S$, and a differentiable function $V \colon \mathbb{R}^n \to \mathbb{R}$ such that:

i. $V(\tilde{x}) = 0$

ii. $V(x) > 0, \ \forall x \in S, x \neq \tilde{x}$

iii. $V(f(x)) - V(x) \leq 0, \ \forall x \in S$

$\Longrightarrow$ Then the equilibrium $\tilde{x}$ is **stable**.

If additionally we have

iv. $V(f(x)) - V(x) < 0, \ \forall x \in S, x \neq \tilde{x}$

$\Longrightarrow$ Then the equilibrium $\tilde{x}$ is **locally asymptotically stable**.

If additionally we have

v. The open set is the whole space $S = \mathbb{R}^n$

vi. $\|x\| \to \infty \implies V(x) \to \infty$

$\Longrightarrow$ Then the equilibrium $\tilde{x}$ is **globally asymptotically stable**.

(Note that conditions iii. and iv. are the main differences with respect to Lyapunov second method for continuous-time.)

Now we will use these methods to analyze the stability of a nonlinear **discrete-time** system. Consider the following system for $\alpha > 0$:

$$\begin{aligned} x_1(k+1) &= \tfrac{\alpha}{1+x_1(k)^2+x_2(k)^2} x_2(k) \\ x_2(k+1) &= \tfrac{\alpha}{1+x_1(k)^2+x_2(k)^2} x_1(k) \end{aligned}$$

1. What are the equilibrium points for the following cases?

   (a) $\alpha \geq 1$
   Hint: Show that $\alpha = (1 + \tilde{x}_1^2 + \tilde{x}_2^2)$ for any non-trivial equilibrium point $(\tilde{x}_1, \tilde{x}_2) \neq (0, 0)$. Then use it to find these non-trivial equilibrium points.

   (b) $\alpha < 1$

2. Using the **Lyapunov <u>first</u> method for discrete-time** systems, for which values of α is the system **locally asymptotically stable** around the equilibrium $(0, 0)$?

3. Using the **Lyapunov <u>second</u> method for discrete-time** systems and considering the Lyapunov function $V((x_1(k), x_2(k))) = x_1(k)^2 + x_2(k)^2$, what can you say about the **global asymptotic stability** of the equilibrium $(0, 0)$ for $\alpha = 1$?

Exercise 4

1	2	3	4	5	Exercise
3	5	4	7	6	25 Points

Consider the linear time invariant system Σ given by the transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\omega_0^2}{(s-\sigma)^2 + \omega_0^2}, \tag{1}$$

where $\omega_0 > 0, \sigma$ are the system parameters.

1. What are the poles of the system? For what conditions on ω_0 and σ is the system Σ given in (1) asymptotically stable?

2. Given the transfer function $G(s)$ in (1) for the system Σ, find the corresponding second order ODE, which describes the output trajectory $y(t)$ given the input signal $u(t)$. Assume that all initial conditions are 0.

3. Setting states to be $x_1(t) = \dot{y}(t)$ and $x_2(t) = y(t)$, write down the state-space representation A, B, C, D of Σ.

4. Set $\omega_0 = 1, \sigma = -1$.
   (a) Compute $|G(j\omega)|$ for $\omega \in \mathbb{R}$.
   (b) Compute $\angle G(j\omega)$ for $\omega \in \mathbb{R}$.
   (c) Compute the steady-state output response of the system given the input $u(t) = \sin(t)$.
   Hint: By using the results from subtasks (a) and (b), you can obtain the output response directly.

5. Consider closed-loop system Σ connected to a controller as shown in Figure 4.

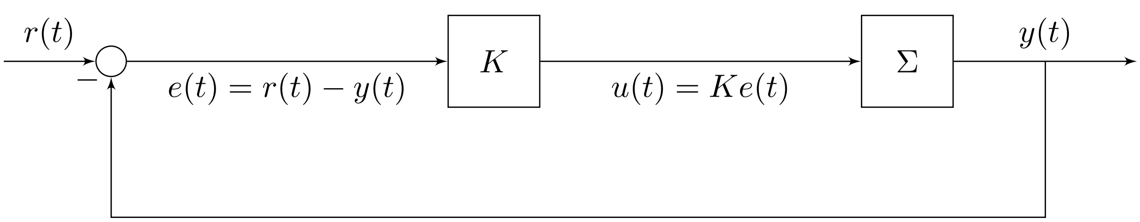

Figure 4: Feedback system

   (a) Write down $G_1(s) = \frac{E(s)}{R(s)}$ of the closed-loop system, where $E(s)$ and $R(s)$ are the Laplace transform of $e(t)$ and $r(t)$ respectively.
   (b) For $\sigma = 0$, compute the steady-state error $\lim_{t\to\infty} e(t)$ given $K \in \mathbb{R}$ and step reference $r(t) = \begin{cases} 1, \ t > 0 \\ 0, \ t < 0 \end{cases}$.
   Is it possible to find such K so that steady-state error equals 0?
   Hint: Use Final Value Theorem.