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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4	5	Exercise
7	3	7	5	3	25 Points

You and a friend have been recruited by the Swiss National Circus to perform a flying trapeze act. Because of your impressive performance in the "Signals and Systems II" final exam, you were asked to analyze the physics of the act, as shown in Figure 1.

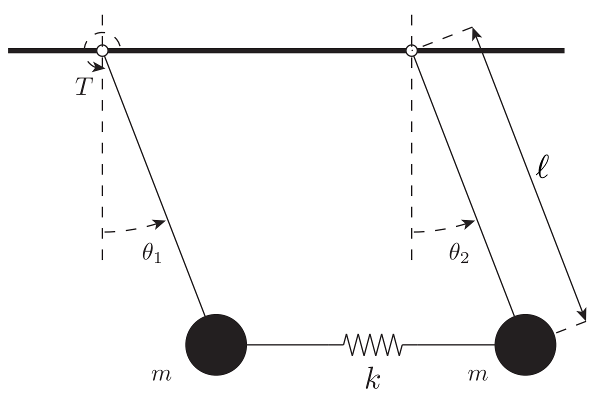

Figure 1: Circus act

In Figure 1, you (left) and your friend (right) are assumed to be point masses of mass m, each attached to massless rods of length ℓ. You are connected by a spring with constant k. Furthermore, you are able to apply a torque $T(t)$ to the rod on the left. The angles that you and your friend make with respect to the vertical axis are denoted by $\theta_1(t)$ and $\theta_2(t)$ respectively. When $\theta_1(t) = \theta_2(t) = 0$, the spring is relaxed. The only forces acting are the gravitational force, spring force, and the torque. The spring force can be assumed to act perpendicular to gravity at all times and proportional to the difference of $\theta_1(t)$ and $\theta_2(t)$. Assume that $k = m\ell$ and $g = \ell$ where g is the gravitational acceleration.

1. Using the laws of motion, derive the dynamics of you and your friend as an equation of the form
$$\dot{x}(t) = f(x(t), u(t)),$$
where $x(t) = \begin{bmatrix}\theta_1(t) & \theta_2(t) & \dot{\theta}_1(t) & \dot{\theta}_2(t)\end{bmatrix}^T$ and $u(t) = T(t)$.
Hint: Use the force balance equation $\Sigma F = I\ddot{\theta}$. The moment of inertia of a point mass m attached to a massless rod of length ℓ is given by $I = m\ell^2$.

2. Show that $x(t) = 0$ is an equilibrium for the dynamics if $T(t) = 0$ for all $t \geq 0$.

3. Linearize the dynamics about the equilibrium in Part 2 using the assumptions that for a small angle θ, we can approximate $\sin(\theta) \approx \theta$ and $\cos(\theta) \approx 1$. Use the same state and input vectors as in Part 1. Assume that during the circus act you will be blindfolded and can only measure your own angle $y(t) = \theta_1(t)$. Show that the linearized dynamics are given by
$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m\ell^2} \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t)\end{aligned}$$

4. Show that the linearized system is observable.

5. At the end of the circus act, you are supposed to release your friend causing him to fly through a hoop. Your friend has not taken "Signals and Systems II" and does not trust that you will be able to release him at the proper angle such that he survives the jump, because you are blindfolded. Explain to your friend why the measurement of your own angle is enough.

Exercise 2

1	2	3	4	5	Exercise
3	5	6	6	5	25 Points

Consider the following system with $x(t) \in \mathbb{R}^3$, $u_1(t) \in \mathbb{R}$, $u_2(t) \in \mathbb{R}$:
$$\dot{x}(t) = \underbrace{\begin{bmatrix} 3 & 0 & -1 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix}}_{B_1} u_1(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{B_2} u_2(t). \tag{1}$$

1. For what values of α is the system asymptotically stable?

2. Assume only input $u_1(t)$ is applied to the system (i.e. $u_2(t) = 0$ for all t). For what values of the parameter β is the system controllable? Provide the controllability matrix and justify your answer.

3. Assume now that $\alpha = -1$, $\beta = -1$ and only $u_1(t)$ is applied to the system. Is it possible to find a gain matrix $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$ such that the closed-loop system under the input $u_1(t) = Kx(t)$,
$$\dot{x}(t) = (A + B_1 K)x(t),$$
is asymptotically stable?

4. Assume now that only input $u_2(t)$ is applied to the system (i.e. $u_1(t) = 0$ for all t). For what values of the parameters α is the system controllable? Provide the controllability matrix and justify your answer.

5. Consider again the case when $\alpha = -1$, $\beta = -1$ and assume now that both the inputs $u_1(t)$, $u_2(t)$ are applied simultaneously. For this system, your boss claims that there exists control inputs to drive the states from $x(0) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\mathsf{T}$ to the origin $x(t_f) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\mathsf{T}$ in $t_f = 1$ s. Do you agree with him?

Exercise 3

1	**2**	**3**	**4**	**Exercise**
5	**9**	**3**	**8**	**25 Points**

Consider the system $\dot{x}(t) = f(x(t))$ defined by

$$\dot{x}_1(t) = x_2(t) \tag{2a}$$
$$\dot{x}_2(t) = -x_1(t) + \frac{1}{3}x_1(t)^3 - x_2(t) \tag{2b}$$

1. Find the equilibria of (2) and determine their stability using the linearization method.

2. Assume that $(0, 0)$ is one of the stable equilibria. To estimate its region of attraction, we define the function $V : \mathbb{R}^2 \to \mathbb{R}$
$$V(x) = \frac{3}{4}x_1^2 - \frac{1}{12}x_1^4 + \frac{1}{2}x_1x_2 + \frac{1}{2}x_2^2. \tag{3}$$
Argue why $V(x)$ is a suitable function to study the region of attraction of $(0, 0)$ using Figure 2. Take the derivative $\frac{d}{dt}V(x(t))$ along the system dynamics and show that it is negative as long as $|x_1| < \sqrt{3}$. Hence argue that the domain of attraction of $(0, 0)$ contains the level set
$$S = \left\{x \in \mathbb{R}^2 \middle| V(x) \leq c\right\},$$
for all $c < 1$.

3. Is $(0, 0)$ globally asymptotically stable? Justify your answer.

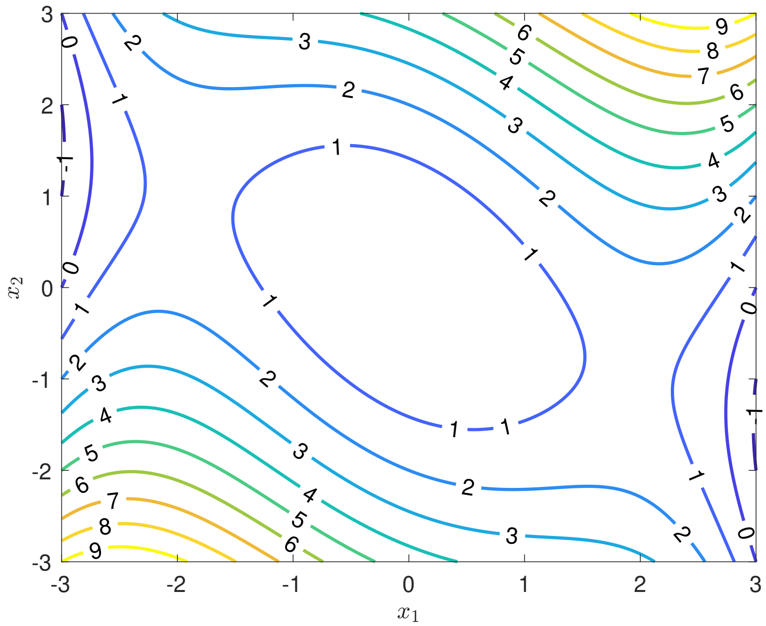

Figure 2: Level sets of the function V

4. Discretize the nonlinear system using the forward Euler method

$$x_{k+1} = x_k + \delta f(x_k),$$

where $\delta > 0$ is a small discretization time step. By assuming $\|x_k\|$ is small, approximate the discretization by a discrete time linear system

$$x_{k+1} = x_k + \delta A x_k.$$

Show that for $\delta = 1/2$ the linearized discretization correctly predicts that $x_k = (0,0)$ is an asymptotically stable equilibrium. Can you also distinguish whether the equilibrium at $(0,0)$ is locally or globally asymptotically stable from the linearized discretization?

Exercise 4

1	2	3	4	5	6	Exercise
4	7	5	5	2	2	25 Points

The simplest possible model of a musical instrument is that of a single resonance. Such a resonance can be modeled for instance as a mass m connected to a damper with parameter R and a spring with parameter k as shown in Figure 3, leading to

$$m\ddot{y}(t) + R\dot{y}(t) + ky(t) = F(t),$$

where $y(t)$ is the horizontal position of the mass measured from its equilibrium for $F(t) = 0$.

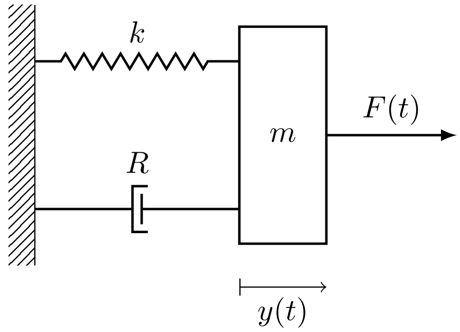

Figure 3: Damped harmonic oscillator

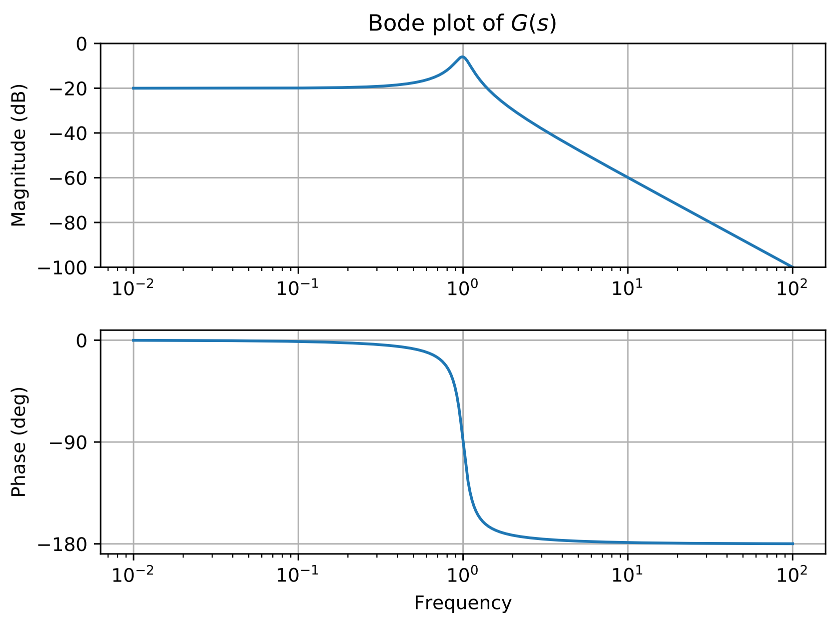

Figure 4: Frequency response of the system with the given constants

1. Using $y(t)$ and $\dot{y}(t)$ as states and $u(t) = F(t)$ as input, write the system in state-space form.

2. Compute the transfer function $G(s)$ of the system. Are there any pole-zero cancellations? What can you conclude about the controllability and observability properties of the system?

3. Let now $R = 2, k = 10, m = 10$. By bringing the system in standard resonance form argue that the frequency response exhibits a maximum and estimate its peak.

4. Given the Bode plot of the system in Figure 4, provide a rough sketch of the Nyquist plot. Include the unit circle in your plot.

5. Estimate the gain and phase margins of the system.

6. Do you think that the system remains stable for arbitrary gain and phase perturbations? Justify your answer.