Automatic Control Laboratory
ETH Zurich
Prof. J. Lygeros

D-ITET
Summer 2017
17.08.2017

Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4	5	Exercise
4	6	4	6	5	25 Points

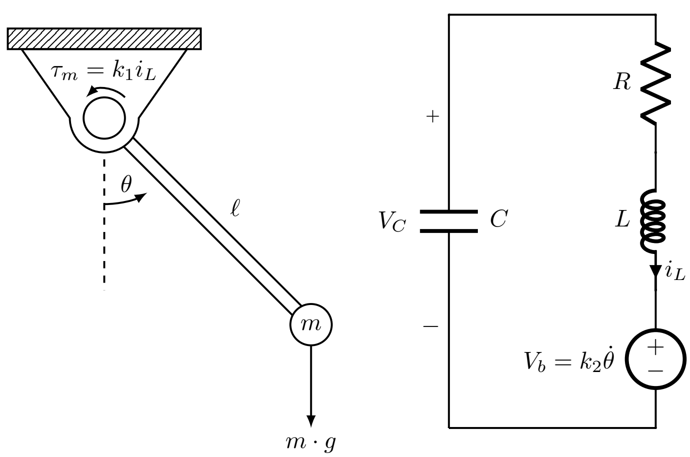

Figure 1: Electromechanical system for Exercise 1

Consider the electromechanical system shown in Figure 1. A motor drives a pendulum with a point-mass m attached to a massless rod of length ℓ (left part of the figure). The motor provides a counterclockwise torque $\tau_m = k_1 i_L$ to the pendulum, where i_L is the current in the motor windings (right part of the figure). The motion of the pendulum generates a voltage $V_b = k_2\dot{\theta}$ in the motor, where $\dot{\theta}$ is the angular velocity of the pendulum. Assume that all joints are frictionless and that there is no aerodynamic drag, so that the only influences acting on the pendulum are the motor torque and gravity.

1. Propose a logical choice for the system state vector $x(t)$. Explain your selection.

2. Write a model of the system in state-space form $\frac{dx(t)}{dt} = f(x(t))$.

Hint: The moment of inertia I of a point mass m which is a distance ℓ from a given axis is $I = m\ell^2$.

3. By using the small angle approximation $\sin(\theta) \approx \theta$, approximate the state-space description of the system by a linear system of the form $\dot{x}(t) = Ax(t)$.

4. Assume that the system parameters are such that in Part 3

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & k_2 \\ 0 & 0 & 0 & 1 \\ 0 & k_1 & -1 & 0 \end{bmatrix}.$$

Compute the characteristic polynomial of the matrix A. For what values of k_1 and k_2 is the system stable?

Hint: According to the Routh-Hurwitz Stability Criterion, the quartic polynomial $\lambda^4 + a_2\lambda^2 + 1$ has a root with a positive real part if and only if $a_2 < 2$.

5. Assume that k_1 and k_2 are chosen so that the system is stable and that the matrix A in Part 4 corresponds to $R = 0$. What would you expect the system behavior to be like qualitatively? What qualitative change would you expect if you consider $R > 0$ instead?

Exercise 2

1	2	3	4	5	Exercise
4	6	5	6	4	25 Points

Consider the linear, time-invariant system

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t), \quad y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x(t). \tag{1}$$

1. Consider a change of coordinates $\hat{x}(t) = Sx(t)$ where S is an invertible matrix. Show that the evolution of $\hat{x}(t)$ is also governed by a linear, time-invariant system

   $$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \quad y(t) = \hat{C}\hat{x}(t)$$

   and provide formulas for $\hat{A}$, $\hat{B}$, and $\hat{C}$ as a function of S, A, B, and C.

2. Find a matrix S such that in the new coordinates the matrix $\hat{A}$ is diagonal. Using S, compute the state transition matrix e^{At} of the original system.

3. Compute the output impulse response $K(t)$ of the system in the original coordinates $x(t)$. Would you expect the answer to be different if the coordinates $\hat{x}(t)$ from Part 2 were used instead?

4. For the system in the original coordinates, find a solution $Q = Q^T$ to the Lyapunov equation $A^TQ + QA = -R$ for

   $$R = \begin{bmatrix} 24 & 0 \\ 0 & 12 \end{bmatrix}.$$

   Based on the solution, argue about the stability of the system.

5. If the Lyapunov equation $\hat{A}^T\hat{Q} + \hat{Q}\hat{A} = -R$ in the coordinates of Part 2 was solved for the same R as in Part 4, would you expect the solution, $\hat{Q}$, to be the same as the solution Q in Part 4? What properties would you expect $\hat{Q}$ to have?

Exercise 3

1	2	3	4	5	Exercise
5	6	4	6	4	25 Points

Consider the continuous-time linear system

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}}_{\bar{A}} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\bar{B}} u(t).$$

1. Show that the state transition matrix is given by

$$e^{\bar{A}t} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t}. \end{bmatrix}$$

   Is the system stable? Is it asymptotically stable? Is it controllable?

2. The system is digitized by sampling the state every T seconds and applying a discrete-time input u_k, for $k = 0, 1, \ldots$ through a zero order hold, giving rise to a discrete-time system

$$x_{k+1} = Ax_k + Bu_k.$$

   Compute the matrices A and B as a function of the sampling time T.

3. For which values of T is the resulting discrete-time system asymptotically stable? For which is it controllable?

4. Your friend from EPFL is too lazy to compute $e^{\bar{A}t}$ and instead proposes to approximate the sampled data system by

$$x\big((k+1)T\big) \approx x(kT) + T\dot{x}(kT),$$

   giving rise to a different discrete time system

$$x_{k+1} = \tilde{A}x_k + \tilde{B}u_k.$$

   Determine the matrices $\tilde{A}$ and $\tilde{B}$. For which values of T is the resulting discrete-time system asymptotically stable? For which is it controllable?

5. Your friend insists that his discrete-time system is better because for some values of T the matrix $\tilde{A}$ is nilpotent. Is this true? Argue that in this case the resulting discrete-time approximation may be misleading.

Exercise 4

1	2	3	4	5	Exercise
3	4	5	7	6	25 Points

Consider the linear system

$$\Sigma_1 : \quad \begin{aligned} \dot{x}_1(t) &= A\,x_1(t) + B\,u_1(t) \\ y_1(t) &= C\,x_1(t) + D\,u_1(t) \end{aligned}$$

given by the matrices

$$A = \begin{bmatrix} -1 & 0 \\ 2 & a \end{bmatrix}\ ,\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\ ,\ C = \begin{bmatrix} 0 & 1 \end{bmatrix}\ ,\ D = 0 \tag{2}$$

where $a \in \mathbb{R}$ is a system parameter.

1. Compute the transfer function $G_1(s)$ of the system Σ_1.

2. Based only on your answer in Part 1, determine whether the system is controllable and observable. For which values of the parameter a is the system asymptotically stable? For which values is it stable? For which values is it unstable? Do not argue by computing the eigenvalues of A, the controllability and observability matrices, nor the Kalman decomposition.

3. Consider the case $a = -1$. By computing the value ω^* for which $\|G_1(j\omega^*)\| = 1$, determine the phase margin of the transfer function $G_1(s)$. Which of the two Bode diagrams (choose (a) or (b)) shown in Figure 2 could correspond to $G_1(s)$ with $a = -1$?

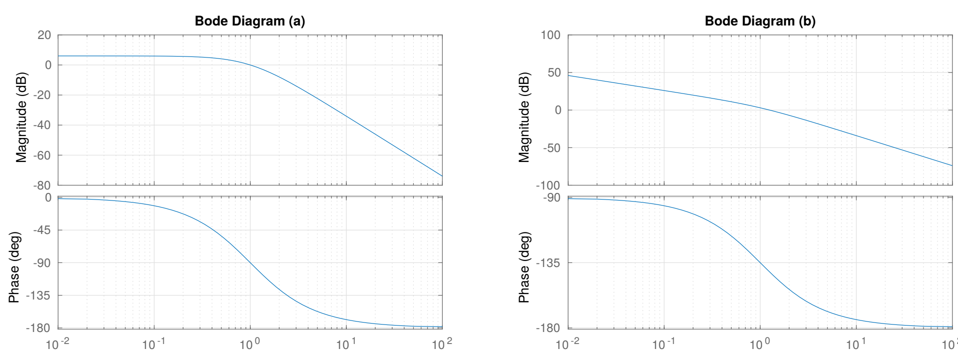

Figure 2: Bode diagram of the open-loop transfer function of Σ_1 for different values of a.

Next, consider the system Σ_1 connected to a controller Σ_2 as shown in Figure 3.

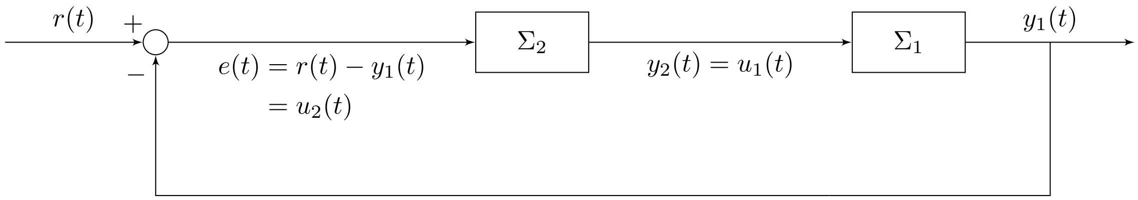

Figure 3: Feedback system

4. Consider the case $a = \frac{1}{4}$ and the controller $\Sigma_2 : y_2(t) = Ke(t)$. Figure 4 shows the Nyquist diagram of the open-loop system Σ_1. For which values of $K \in \mathbb{R}$ is the closed-loop system shown in Figure 3 stable?

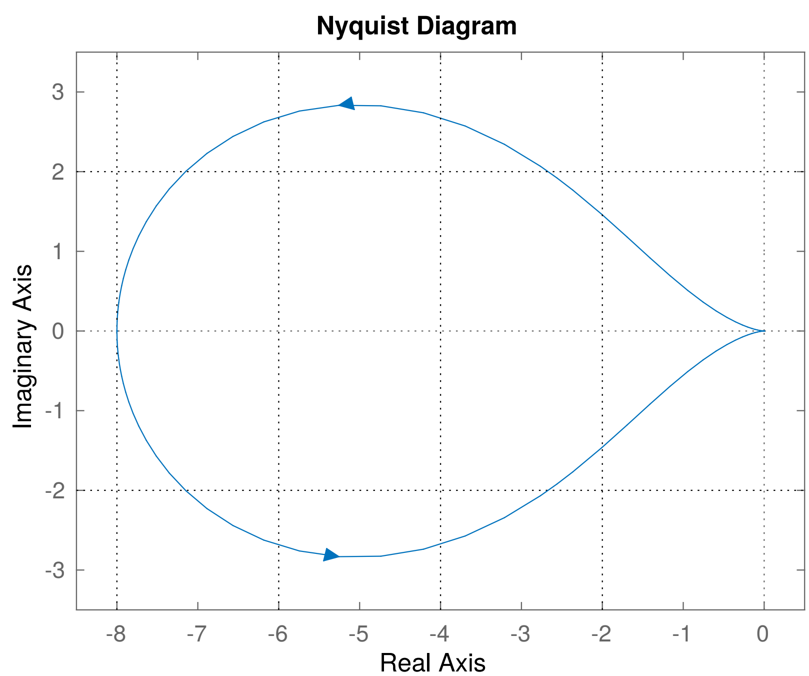

Figure 4: Nyquist diagram of the open-loop transfer function of Σ_1.

5. Based on your answer in Part 4 you decide to use the controller $\Sigma_2 : y_2(t) = Ke(t)$ with $K = 1$. Your friend from EPFL insists that the controller Σ_2 given by the transfer function

$$G_2(s) = \frac{1}{2}\frac{s - \frac{1}{4}}{s + 3}$$

will work better than yours. Compute the transfer function from the input $r(t)$ to the output $y_1(t)$ for the block diagram shown in Figure 3 for $G_1(s)$ with $a = \frac{1}{4}$ combined with the $G_2(s)$ proposed by your friend. Is the system stable? Explain to your friend why using his controller is nonetheless a bad idea.