Automatic Control Laboratory
ETH Zurich
Prof. J. Lygeros

D-ITET
Summer 2016
17.08.2016

# Signal and System Theory II

**This sheet is provided to you for ease of reference only.**
***Do not* write your solutions here.**

## Exercise 1

| 1 | 2 | 3 | 4 | 5 | Exercise |
|---|---|---|---|---|---|
| 5 | 6 | 3 | 3 | 8 | 25 Points |

Consider the system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix},\ B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D = 1,$$

and $a, b_1, b_2 \in \mathbb{R}$.

1. Is $A$ diagonalizable? If so, write it in the form $A = W\Lambda W^{-1}$. If not, explain why not.
2. Compute the matrix exponential $e^{At}$. Simplify your answer until all entries of the matrix are real.
3. For what values of $a$ is the system stable? For what values of $a$ is the system asymptotically stable?
4. For what values of $b_1$ and $b_2$ is the system controllable?
5. Now let $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

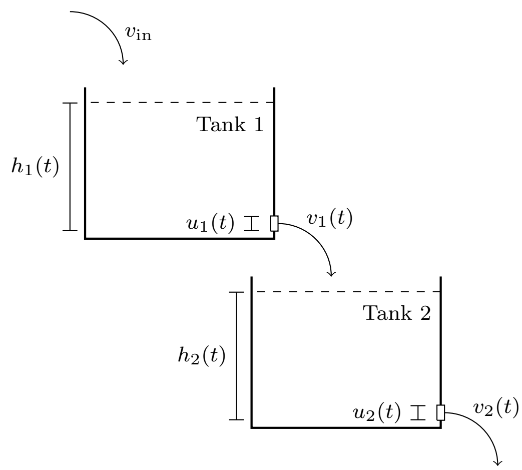
   Compute the transfer function of the system. For the initial state $x_0 = [0\ 0]^T$, compute the response of the system, for all times $t \geq 0$, to a unit step input

   $$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0\ . \end{cases}$$

# Exercise 2

| 1 | 2 | 3 | 4 | 5 | Exercise |
|---|---|---|---|---|---|
| 6 | 3 | 6 | 4 | 6 | 25 Points |

Consider the system of two water tanks shown in the figure.




Tank $i \in \{1, 2\}$ has constant surface area $A_i$ and a controllable outflow area $u_i(t)$. There is a fixed stream of water $v_{\text{in}}$ (in $m^3/s$) flowing into the top tank. The rate at which water flows out of tank $i \in \{1, 2\}$ (in $m^3/s$) is given by

$$v_i(t) = C_v u_i(t)\sqrt{2gh_i(t)}$$

where $C_v > 0, g > 0$ are constants and $h_i(t)$ is the height of water in tank $i$.

1. Using $x(t) = \begin{bmatrix} h_1(t) & h_2(t) \end{bmatrix}^T$ as the state and $u(t) = \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^T$ as input, write the system in state-space form

$$\frac{dx(t)}{dt} = f(x(t), u(t)).$$

2. Given constants $\alpha_1, \alpha_2 > 0$, find the constant inputs $\bar{u}$ that maintain the state at $\bar{x} = \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix}^T$.

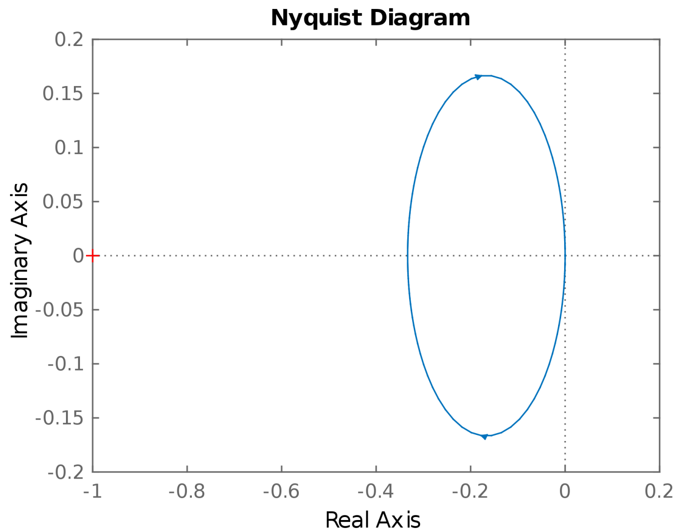
3. Linearize the system around the steady state computed in the previous task, that is compute the matrices $A$ and $B$ in

$$\frac{d}{dt}(\Delta x(t)) = \underbrace{\begin{bmatrix} \frac{df_1(x,u)}{dx_1} & \frac{df_1(x,u)}{dx_2} \\ \frac{df_2(x,u)}{dx_1} & \frac{df_2(x,u)}{dx_2} \end{bmatrix}\Bigg|_{\substack{x=\bar{x} \\ u=\bar{u}}}}_{A} \Delta x(t) + \underbrace{\begin{bmatrix} \frac{df_1(x,u)}{du_1} & \frac{df_1(x,u)}{du_2} \\ \frac{df_2(x,u)}{du_1} & \frac{df_2(x,u)}{du_2} \end{bmatrix}\Bigg|_{\substack{x=\bar{x} \\ u=\bar{u}}}}_{B} \Delta u(t)$$

where $\Delta x(t) = x(t) - \bar{x}$ and $\Delta u(t) = u(t) - \bar{u}$.

4. Is the linearized system stable? What, if anything, can you infer about the stability of the nonlinear system?

5. The following plot shows the Nyquist diagram for the transfer function from the first input $\Delta u_1(t)$ to the first state $\Delta x_1(t)$ of the linearized system.




You want to introduce a feedback loop $\Delta u_1(t) = r(t) - K\Delta x_1(t)$ for some $K \in \mathbb{R}$. For which values of $K$ will the stability properties of the closed loop system from $r(t)$ to $\Delta x_1(t)$ differ from those of the open loop system from $\Delta u_1(t)$ to $\Delta x_1(t)$? *Hint: The intersections of the curve with the real axis are at* $-\frac{1}{3}$ *and* $0$.

# Exercise 3

| 1 | 2 | 3 | 4 | 5 | Exercise |
|---|---|---|---|---|---|
| 5 | 3 | 5 | 5 | 7 | 25 Points |

Consider the discrete-time system given by the $N$-th order difference equation

$$w_k + a_1 w_{k-1} + a_2 w_{k-2} + \cdots + a_N w_{k-N} \; = \; b_0 u_k \tag{1}$$

for $k = N, N+1, \ldots$ with $w_k \in \mathbb{R}$, $u_k \in \mathbb{R}$.

1. Transform (1) into the state space representation (2) using the state vector $x_k = [w_{k-N} \; w_{k-(N-1)} \; \ldots \; w_{k-1}]^\top$, the input $u_k$ and the output $y_k = w_{k-1}$. State the resulting system matrices $A$, $B$, $C$ and $D$.

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned} \tag{2}$$

For the remaining parts of this exercise, consider the special case

$$w_k - \frac{3}{4} w_{k-1} + \frac{1}{8} w_{k-2} \; = \; 2u_k \tag{3}$$

2. Transform (3) into the state space representation (2) using the state vector $x_k = [w_{k-2} \; w_{k-1}]^\top$, the input $u_k$ and the output $y_k = w_{k-1}$. State the resulting system matrices $A$, $B$, $C$ and $D$.

3. Determine the stability of system (3) with $u_k = 0$ for all $k$. Is the system controllable? Is it observable?

4. Starting from an arbitrary initial condition $x_N \in \mathbb{R}^2$ with $u_k = 0$ for all $k$, does the state $x_k$ of the system (3) reach $[0 \; 0]^\top$ in a finite number of steps and remain there? There is no need to compute the response, but you do need to justify your answer.

5. Is there a state feedback law $u_k = Fx_k$ for system (3) such that, starting from any initial condition $x_N \in \mathbb{R}^2$, the state reaches $[0 \; 0]^\top$ in a finite number of steps and remains there? If yes, state the dimension of $F$ and compute its entries. If no, explain why this is not possible.

# Exercise 4

| 1 | 2 | 3 | 4 | 5 | Exercise |
|---|---|---|---|---|---|
| 7 | 5 | 5 | 3 | 5 | 25 Points |

Consider the following mass-spring-damper system

$$m\,\ddot{x}(t) = -c(x(t)) - d(\dot{x}(t)), \tag{4}$$

with
$x(t) \in \mathbb{R}$: the position of the mass $m$,
$c(\cdot) : \mathbb{R} \mapsto \mathbb{R}$: the spring force acting on the mass $m$,
$d(\cdot) : \mathbb{R} \mapsto \mathbb{R}$: the damping force acting on the mass $m$.

For the spring force $c(x)$ and damper force $d(\dot{x})$, the following hold:

$$x\,c(x) > 0, \quad \text{for all} \quad x \neq 0, \tag{5}$$
$$c(0) = 0, \tag{6}$$
$$\dot{x}\,d(\dot{x}) > 0, \quad \text{for all} \quad \dot{x} \neq 0, \tag{7}$$
$$d(0) = 0, \tag{8}$$

1. (a) Write the system in state-space form $[\dot{x}_1(t)\ \dot{x}_2(t)]^T = f(x_1(t), x_2(t))$, with $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$.
   (b) Is the system linear?
   (c) Is the system time invariant?
   (d) Is the system autonomous?
2. Compute all equilibrium points of the system.
3. Using the energy of the system $V(x) = \frac{1}{2}mx_2^2 + \int_0^{x_1} c(s)ds$ as a Lyapunov function, show that the equilibrium at $(0,0)$ is stable. You may assume that $V(x) > 0$ for all $x \neq 0$.
4. Can you use Theorem 7.3 in the Lecture Notes to show that the origin is globally asymptotically stable?
5. Can you use Theorem 7.4 in the Lecture Notes to show that the origin is globally asymptotically stable? You may assume that $c(\cdot)$ is such that for any $\alpha > 0$, the set $S = \{x \in \mathbb{R}^2 | V(x) \leq \alpha\}$ is compact.