Automatic Control Laboratory
ETH Zurich
Prof. J. Lygeros

D-ITET
Examination Summer 2015
15.08.2015

Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	**2**	**Exercise**
12	**13**	**25 Points**

1. Consider the following system

$$\dot{x}(t) = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t) \tag{1}$$
$$y(t) = \begin{pmatrix} 1 & 2 \end{pmatrix} x(t).$$

   (a) Compute the impulse response $K(t)$ of the system (1). Can you draw a conclusion about the controllability and observability of the system without computing the controllability and observability matrices?

   (b) Compute the transfer function of the system (1).

   (c) Compute the zero state response $y(t)$ for the input $u(t) = \begin{cases} 2 \text{ for } t \geq 0 \\ 0 \text{ for } t < 0 \end{cases}$.

2. Consider a dynamical system described by the transfer function

$$Y(s) = \frac{s-1}{(s+1)(s-2)(s+3)} U(s). \tag{2}$$

   (a) Bring the system to its observable canonical form, which we denote by $\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t)$.

   (b) You would like to design an observer for the system (2) in the form

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) + L[y(t) - C\tilde{x}(t)],$$

   where $L \in \mathbb{R}^{3\times 1}$. Derive the differential equation of the dynamics of the error

$$e(t) = x(t) - \tilde{x}(t).$$

   (c) Design now the observer gain L such that the eigenvalues of the error dynamics are at $-1, -2, -3$.

Exercise 2

1	2	3	4	5	Exercise
4	6	5	7	3	25 Points

Consider the discrete-time system

$$
\begin{aligned}
x(k+1) &= \underbrace{\begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ a & 0 & 0.5 \end{pmatrix}}_{A} x(k) + \underbrace{\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}}_{B} u(k), \\
y(k) &= \underbrace{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}}_{C} x(k),
\end{aligned} \tag{3}
$$

where $x(k) \in \mathbb{R}^3$ is the system state, $u(k) \in \mathbb{R}$ the input, and $y(k) \in \mathbb{R}$ the output at time step k, respectively. The constant $a \in \mathbb{R}$ is a parameter of the system.

1. Compute the eigenvalues of the system matrix A. Is the system asymptotically stable?
2. What is the set of reachable states for the system (3) for $a = 0$ and for $a \neq 0$?

   Hint: The computation of reachable states is the same as for continuous-time systems.
3. For which values of a is the system observable?
4. How many poles will the transfer function of (3) have? Consider all possible values of a.

   Hint: Use

   $$\operatorname{rank}\begin{pmatrix} C \\ \lambda I - A \end{pmatrix} \quad \text{and} \quad \operatorname{rank}\begin{pmatrix} B & \lambda I - A \end{pmatrix}$$

   to determine unobservable and uncontrollable states.
5. Your friend from EPFL suggests adding an actuator to system (3) to make the system controllable for all values of a. You can choose to add either

   $$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_2(k) \quad \text{or} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_3(k)$$

   to the state-transition equation (3). Which one should you choose?

Exercise 3

1	2	3	4	5	6	Exercise
4	3	3	5	3	7	25 Points

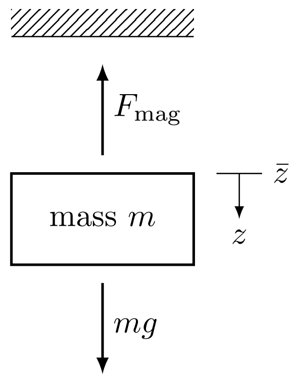

Figure 1: System for Exercise 1

As part of a prototype car project, you are tasked with the design of a suspension system. Your boss likes futuristic designs and asks you to create a magnetic suspension system, as schematically depicted in Figure 1. The wheel part of the system is modeled as a constant magnet of mass m. The force $F_{\text{mag}}\big(z(t), u(t)\big)$ on this part, induced by an electromagnet in the car frame, is modeled as

$$F_{\text{mag}}\big(z(t), u(t)\big) = C\frac{u(t)}{(z(t)+\bar{z})^2}, \tag{4}$$

where $\bar{z} > 0$ is the desired steady-state position, $u(t)$ is the input of the system and $C > 0$ is a physical constant. Assume gravity acts on the wheel.

1. Write down the differential equation that describes the evolution of $z(t)$. Is it linear? Is it time invariant?

2. Find the constant input $\bar{u}$ that makes $\bar{z}$ an equilibrium, i.e. holds the wheel at $z(t) = 0$.

3. Using $x(t) = \begin{pmatrix} z(t) & \dot{z}(t) \end{pmatrix}^T$ as state and $y(t) = z(t)$ as output, write the system in state space form:

$$\begin{aligned} \dot{x}(t) &= f\big(x(t), u(t)\big) \\ y(t) &= h\big(x(t), u(t)\big) \end{aligned}$$

4. Linearize the dynamics about the equilibrium $x_1(t) = 0, x_2(t) = 0, u(t) = \bar{u}$, i.e. compute the matrices in the equation

$$\dot{\delta x}(t) = \frac{\partial f}{\partial x}(\bar{x}, \bar{u})\delta x(t) + \frac{\partial f}{\partial u}(\bar{x}, \bar{u})\delta u(t) \tag{5}$$

   where $\delta x(t) = x(t) - \bar{x}$ and $\delta u(t) = u(t) - \bar{u}$.

5. Investigate the stability of the linearized system. Do you think the suspension will work with just a constant input $u(t) = \bar{u}$?

6. To stabilize the system, you decide to apply state feedback of the form

$$u(t) = \bar{u} + k_1 x(t) + k_2 x_2(t). \tag{6}$$

   Use linearization to determine for which values of k_1 and k_2 the equilibrium $x_1(t) = 0, x_2(t) = 0, u(t) = \bar{u}$ will be locally asymptotically stable. (Hint: Use $\delta u(t) = k_1 x_1(t) + k_2 x_2(t)$ and your linearized system from Task 4.)

Exercise 4

1	2	3	4	Exercise
3	9	8	5	25 Points

Consider the continuous time system

$$\dot{z}(t) = az(t) - u(t), \tag{7}$$

with state $z(t) \in \mathbb{R}$, input $u(t) \in \mathbb{R}$ and constant parameter $a \in \mathbb{R}$.

1. Is the system linear? Is it time invariant? Discuss the stability properties of system (7) as a function of a when $u(t) = 0$ for all t.

2. Consider now the dynamic feedback controller

$$\begin{aligned} u(t) =& k(t)z(t) \\ \dot{k}(t) =& z^2(t). \end{aligned} \tag{8}$$

   (a) Write the dynamics of the closed loop system using $x(t) = (z(t)\,k(t))^\top$ as the state, that is,

$$\dot{x}(t) = \begin{pmatrix} \dot{z}(t) \\ \dot{k}(t) \end{pmatrix} = \begin{pmatrix} f_z\,(z(t), k(t), a) \\ f_k\,(z(t), k(t), a) \end{pmatrix} \tag{9}$$

   (b) Is the closed loop system (9) linear? Is it time-invariant?

   (c) Compute all equilibrium points of system (9).

   (d) What can you conclude about the stability properties of these equilibrium points from the linearization technique?

3. Consider the Lyapunov function $V_b(z,k) := \frac{1}{2}z^2 + \frac{1}{2}(k-b)^2$, for some $b \in \mathbb{R}$.

   (a) Compute the Lie derivative of $V_b(z,k)$ along the dynamics given in (9).

   (b) Discuss what conclusion you can draw about the stability of the equilibrium $\hat{x} = (0\ b)^\top$ for the cases $b < a$, $b = a$ and $b > a$.

4. (a) Intuitively, what do you think will happen to $z(t)$ as $t \to \infty$?

   (b) Suppose $a > 0$. Sketch the dynamics of system (9) in the plane (k, z). To this end, consider the sign of the derivatives along the axes $z = 0$ and $k = a$ and the value of the Lie derivative for $b = a$.