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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4	5	Exercise
4	5	4	7	5	25 Points

Consider the following system:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \text{where}$$
$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha^2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \alpha + 1 \end{bmatrix} \text{ and } \alpha \in \mathbb{R}.$$

1. Determine the eigenvalues and eigenvectors of A for all values of α.

2. Show that for $\alpha < 0$ the matrix exponential e^{At} is given by

$$e^{At} = \frac{1}{\alpha^2 - \alpha} \begin{bmatrix} (\alpha^2 - \alpha)e^{\alpha t} & e^{\alpha^2 t} - e^{\alpha t} \\ 0 & (\alpha^2 - \alpha)e^{\alpha^2 t} \end{bmatrix}.$$

   Compute also the matrix exponential e^{At} for $\alpha = 0$.

3. For which values of $\alpha \in \mathbb{R}$ is the system stable?

4. Let $\alpha = 0$ and $x(0) = x_0 = [0\ 0]^\top$. Is it possible to find an input $u(t)$ which drives the system to $x(1) = [1\ 0]^\top$? If your answer is yes, provide such an input. If your answer is no, prove that such an input cannot exist.

5. Let $\alpha = -1$ and $x(0) = x_0 = [1\ 0]^\top$. Is it possible to find an input $u(t)$ which drives the system to $x(1) = [e^{-1}\ 0]^\top$? If your answer is yes, provide such an input. If your answer is no, prove that such an input cannot exist.

Exercise 2

1	2	3	4	Exercise
9	6	4	6	25 Points

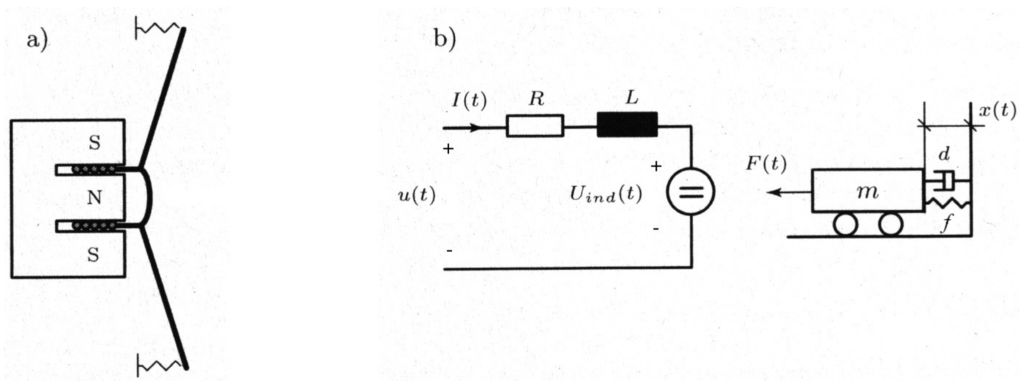

Figure 1: Simplified loudspeaker, a) sketch, b) electromechanical model.

In this task you will model a loudspeaker, sketched in Figure 1. The loudspeaker consists of an electrical and a mechanical part:

- the electrical part is represented by an RL-network with resistance R, inductance L, and an additional self-inductance voltage $U_{ind}(t)$;
- the mechanical part is modeled as a linear damped mass-spring system with mass m, damper constant d, and spring constant f, on which the force $F(t)$ is applied.

The two subsystems are coupled by the coil, which has diameter D, n windings, and is immersed in a homogeneous permanent magnetic field of strength B. A current I through the coil will produce a force $F(t) = B \cdot n \cdot D \cdot \pi \cdot I(t) = \kappa \cdot I(t)$, where we have set $\kappa = B \cdot n \cdot D \cdot \pi$. As the coil oscillates in the magnetic field, the induced voltage in the coil is $U_{ind}(t) = B \cdot n \cdot D \cdot \pi \cdot v(t) = \kappa \cdot v(t)$, where $v(t)$ is the velocity of the mass m.

1. Derive the equations representing this system and bring them into state space form. Use $\tilde{x}(t) = [I(t)\ \ x(t)\ \ v(t)]^\top$ as the state vector, $\tilde{u}(t) = u(t)$ as the input and $\tilde{y}(t) = x(t)$ as the output.

2. Assume $L = m = f = d = \kappa = 1$ and $R = 2$ (with the respective SI-units). Is the system stable? Is it controllable? Is it observable?

   If you could not solve part 1, use $A = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$.

   Hint: there is an eigenvalue $\lambda_1 = -1$.

3. For the parameters given in part 2, derive the transfer function $G(s)$.

   Hint: $A^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{det(A)} \begin{bmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{bmatrix}$

4. Compute the impulse response of the system.

Exercise 3

1	2	3	4	Exercise
6	6	6	7	25 Points

Consider the continuous-time nonlinear system

$$\begin{aligned} \dot{x}(t) &= (x^2(t) + y^2(t) - 1)x(t) - y(t), \\ \dot{y}(t) &= (x^2(t) + y^2(t) - 1)y(t) + x(t), \end{aligned} \tag{1}$$

where $x(t), y(t) \in \mathbb{R}$.

1. This system has a unique equilibrium. What is it? Can you say something about its stability?

2. Consider the change of coordinates

$$x(t) = r(t)\cos(\theta(t)) \tag{2}$$
$$y(t) = r(t)\sin(\theta(t)). \tag{3}$$

   Note that these are the polar coordinates, hence you can assume $r(t) \geq 0$. Show that, in the $r(t), \theta(t)$ coordinates, the system dynamics are given by

$$\begin{aligned} \dot{r}(t) &= r(t)(r^2(t) - 1), \\ \dot{\theta}(t) &= 1. \end{aligned} \tag{4}$$

   Hint: To compute the derivative of r start from the equality $r^2 = x^2 + y^2$. To compute the derivative of θ, differentiate both sides of (3) and then substitute (1), (2) and (3).

3. Compute the sign of $\dot{r}(t)$ when $r(t) > 1, r(t) = 1$, and $r(t) < 1$. Use this information to sketch the evolution of the trajectories of system (1) in the (x, y)-plane. Are there any periodic trajectories?

4. Using the analysis in part 3, suggest a Lyapunov function for the equilibrium found in part 1. You may assume that the set $S = \{(x, y) | x^2 + y^2 < 1\}$ is open and invariant. Use your Lyapunov function to show that the equilibrium point is locally asymptotically stable and to estimate its domain of attraction.

Exercise 4

1	2	3	4	Exercise
7	6	7	5	25 Points

Consider a discrete-time linear system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k) + Du(k), \end{aligned} \tag{5}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ the input, $y(k) \in \mathbb{R}^p$ the output, and the system matrices have appropriate dimensions. For $N \geq n$ assume that noise free measurements of the input $u(k)$ and output $y(k)$ for $k = 0, \dots, N-1$ are stacked in the vectors

$$U = \begin{bmatrix} u(0) \\ \vdots \\ u(N-1) \end{bmatrix} \in \mathbb{R}^{mN}, \quad Y = \begin{bmatrix} y(0) \\ \vdots \\ y(N-1) \end{bmatrix} \in \mathbb{R}^{pN}. \tag{6}$$

1. Assume you want to compute the initial state $x(0)$ of the system (5) from the observed input and output trajectories in (6). Derive a system of linear equations

   $$Mx(0) = q$$

   which allows you to do so. Express M and q in terms of the system matrices A, B, C, and D and the vectors U and Y. What are the dimensions of M and q?

2. Assume $m = p = 1$ and $N = n$. Prove that you can uniquely determine the initial state $x(0)$ from inputs $u(0), \dots, u(N-1)$ and outputs $y(0), \dots, y(N-1)$ *if and only if* the observability matrix

   $$Q := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

   satisfies $\text{rank}(Q) = n$.

3. Assume the values of the system matrices in (5) are

   $$A = \begin{bmatrix} 1 & 1.5 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0.9 \end{bmatrix}, \quad D = 0.$$

   You measure the following values for the input and output of the system:

   $$u(0) = 1, \;\; u(1) = -2, \;\; y(0) = 2.9, \;\; y(1) = 9.9$$

   Compute the initial state $x(0)$ that generated this output trajectory.

4. Give reasons why the system of linear equations from part 1 should generally *not* be used to compute the initial state of a system if the input and output measurements are affected by noise. What alternative would you suggest to reliably estimate the state of a system? What condition should the system fulfill for your suggested method to work?