Automatic Control Laboratory
ETH Zurich
Prof. J. Lygeros

D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4	Exercise
4	7	7	7	25 Points

Consider the linear time-invariant single input, single output discrete-time system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} x(k) + \begin{bmatrix} 0.5 \\ 2 \end{bmatrix} u(k), \\ y(k) &= Cx(k) = \begin{bmatrix} 1 & 2 \end{bmatrix} x(k). \end{aligned} \tag{1}$$

1. Is the system asymptotically stable? Assume $u(k) = 0$ for all $k \in \mathbb{N}$.
2. Compute the transfer function, $G(z)$, of system (1). Are there any pole-zero cancellations? What can you conclude about the controllability and observability properties of system (1)?
3. Suppose that you want to stabilize the system using a feedback control law $u(k) = Kx(k)$, $K \in \mathbb{R}^{1\times 2}$. Derive the matrix A_K of the closed loop system, i.e.

   $$\begin{aligned} x(k+1) &= A_K x(k), \\ y(k) &= Cx(k). \end{aligned}$$

   Find a matrix K such that the closed loop system is asymptotically stable.

   Hint: choose K such that A_K is upper triangular.
4. Consider now an observer that generates an estimate $\hat{x}(k)$ of the state $x(k)$ of system (1), by starting with an arbitrary $\hat{x}(0)$ and evolving according to

   $$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L\left[y(k) - C\hat{x}(k)\right].$$

   Derive a state space model for the evolution of the observation error $e(k) = \hat{x}(k) - x(k)$. Determine the entries of the matrix $L \in \mathbb{R}^{2\times 1}$ so that the observation error goes to zero in a finite number of steps for every initial estimate $\hat{x}(0)$.

   Hint: Try the same trick as in part 3 above.

Exercise 2

1	2	3	4	Exercise
5	6	7	7	25 Points

Consider the following system:

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 3 & -5 \\ 1 & 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t) \qquad (2)$$
$$y(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x(t).$$

1. Is the system stable? Assume $u(t) = 0$ for all $t \geq 0$.

2. Let
$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \widehat{x}(t) = Tx(t).$$
   Derive the matrices $\widehat{A}, \widehat{B}, \widehat{C}$ for the new system
$$\dot{\widehat{x}}(t) = \widehat{A}\widehat{x}(t) + \widehat{B}u(t) \qquad (3)$$
$$y(t) = \widehat{C}\widehat{x}(t)$$
   and show that $\widehat{A}$ is diagonal. What can you infer about the columns of the matrix T^{-1}?

   Hint: T is invertible and
$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

3. Is the new system (3) controllable? Is it observable? Is it stabilizable? Is it detectable? What can you infer about the corresponding properties of the original system (2)?

   Hint: Note the location of zero entries in the matrices $\widehat{B}$ and $\widehat{C}$.

4. Compute the zero input response $y(t)$ of system (2) as a function of the initial state $x(0) = x_0 \in \mathbb{R}^3$. Hence show that $y(t) \to 0$ as $t \to \infty$. How does this relate to your answers in part 1 and part 3 of this exercise?

   Hint: Use your answer to part 2 of this exercise to compute $y(t)$.

Exercise 3

1	2	3	4	Exercise
7	7	7	4	25 Points

Consider the mechanical system given in Figure 1 consisting of two masses coupled with springs and dampers. The springs and dampers are in an equilibrium position if $\xi_1 = \xi_2 = u = 0$. The spring constants are given by k and the damping coefficients are denoted by d.

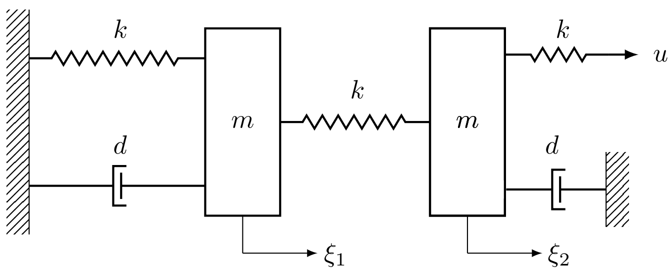

Figure 1: Dynamical System

1. Using $x_1(t) = \xi_1(t), x_2(t) = \dot{\xi}_1(t), x_3(t) = \xi_2(t), x_4(t) = \dot{\xi}_2(t)$ as states, the displacement $u(t)$ as input, and $y(t) = \xi_1(t) - \xi_2(t)$ as output, derive the equations of motion for the system in state space form.

2. Show that if instead one uses $\hat{x}_1(t) = \frac{1}{2}(\xi_1(t)+\xi_2(t)), \hat{x}_2(t) = \frac{1}{2}(\dot{\xi}_1(t)+\dot{\xi}_2(t)), \hat{x}_3(t) = \frac{1}{2}(\xi_1(t) - \xi_2(t))$ and $\hat{x}_4(t) = \frac{1}{2}(\dot{\xi}_1(t) - \dot{\xi}_2(t))$ as states, the state equations take the form

$$\frac{d}{dt}\hat{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\alpha_1 & -\beta_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\alpha_2 & -\beta_2 \end{pmatrix} \hat{x}(t) + \begin{pmatrix} 0 \\ \gamma_1 \\ 0 \\ -\gamma_2 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 0 & 0 & \delta & 0 \end{pmatrix} \hat{x}(t),$$

   for some positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta$. Determine the value of these constants.

3. Show that the system is asymptotically stable.

   Hint: Use your answer to part 2 and the fact that $\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det(A)\det(B)$.

4. Is the system observable? Provide a physical intuition for your answer.

Exercise 4

1	2	3	4	5	Exercise
4	6	6	6	3	25 Points

During your summer internship, your supervisor asks you to characterize the dynamics of an autonomous, time-invariant, analog electrical circuit. In an experiment, you measure the system response for two different initial conditions, which happen to be

$$x(0) = 1: \qquad x(t) = \frac{1}{4e^t - 3}$$
$$x(0) = 2: \qquad x(t) = \frac{2}{7e^t - 6}.$$

1. Is the system linear?
2. After repeating the above experiment for several initial conditions you find out that the system response can be generally described by

$$x(t) = \frac{x_0}{e^t - 3x_0 + 3e^t x_0}, \tag{4}$$

with $x_0 = x(0)$. Assuming that the system has dimension one, find the corresponding system equations, i.e., find a function f such that (4) solves the ODE $\dot{x}(t) = f(x(t))$ with $x(0) = x_0$.

Hint: Differentiate (4), then add and subtract $3x_0^2$ to and from the numerator.

3. Find all equilibria of the system and determine their stability.
4. Sketch $f(x)$ as a function of x and indicate with arrows the direction in which the state moves on the x-axis.
5. How would you design your experiments if you had to show that a system is not time-invariant?