Automatic Control Laboratory
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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	Exercise
10	6	9	25 Points

Consider the following discrete time system

$$y(k+2) + a_1 y(k+1) + a_0 y(k) = b_1 u(k+1) + b_0 u(k), \tag{1}$$

where $u(k) \in \mathbb{R}$ denotes the input, $y(k) \in \mathbb{R}$ denotes the output, and $a_0, a_1, b_0, b_1 \in \mathbb{R}$ are constant coefficients.

1. By defining an appropriate state $x(k) \in \mathbb{R}^2$ show that the system can be put into the observable canonical form

$$\begin{aligned} x(k+1) &= \begin{bmatrix} -a_1 & 1 \\ -a_0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} b_1 \\ b_0 \end{bmatrix} u(k), \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k). \end{aligned}$$

2. Let $b_1 = 2b_0$. Under which conditions is the system controllable? Under which conditions is it observable?

3. Compute the transfer function of the system. For the case where $b_1 = 2b_0$, comment on any pole-zero cancelations that may occur.

Exercise 2

1	2	3	4	Exercise
5	8	7	5	25 Points

Consider the continuous time system

$$\begin{aligned}\dot{x}_1(t) &= -2x_1(t) + 3x_2(t) + u(t)\\ \dot{x}_2(t) &= 4x_2(t) + 2u(t)\\ y(t) &= x_1(t)\end{aligned}$$

where $x(t) = [x_1(t) \;\; x_2(t)]^T$ are the system states, $u(t)$ is the control input, and $y(t)$ is the measurement.

1. Derive matrices A, B, C, and D to bring the system in the standard state space form:

   $$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

   Is the system stable? Is it controllable? Is it observable?

2. Consider a state feedback law $u(t) = Kx(t)$. Is it possible to find a matrix $K \in \mathbb{R}^{1\times 2}$ such that the eigenvalues of the closed loop system are both equal to -4? Comment in relation to your answer in Part (1).

3. An observer constructs an estimate $\tilde{x}(t)$ of the state $x(t)$. Its dynamics are described by the equations

   $$\begin{aligned}\dot{\tilde{x}}(t) &= A\tilde{x}(t) + Bu(t) + L(y(t) - \tilde{y}(t))\\ \tilde{y}(t) &= C\tilde{x}(t) + Du(t) \;\; .\end{aligned}$$

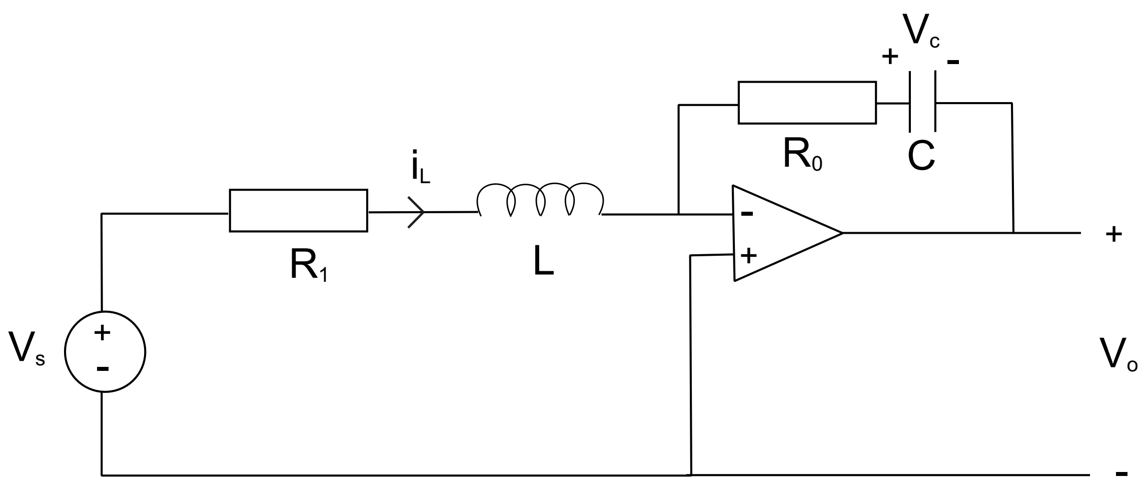
   The estimation error is given by $e(t) = x(t) - \tilde{x}(t)$. Design the entries of the gain matrix $L \in \mathbb{R}^{2\times 1}$ so that the error dynamics have all eigenvalues equal to -1.

4. Compute all states $\hat{x} \in \mathbb{R}^2$ for which there exists a constant input $\hat{u} \in \mathbb{R}$ such that the system is in equilibrium.

Exercise 3

1	2	3	Exercise
10	7	8	25 Points

Consider the following circuit:



1. Write down a state space model for this circuit. Use $x_1 = i_L$, $x_2 = V_c$ as the states of the system, $u = V_s$ as input and $y = V_o$ as output. Assume that the operational amplifier is ideal.

2. Assume that $C = 1F, R_0 = R_1 = 1\Omega, L = 1H$. Calculate the transfer function $G(s)$. Discuss the potential reasons of any pole-zero cancellations.

3. Compute the zero input transition ($V_s(t) = 0$ for all $t \geq 0$) for the initial state $x(0) = [1 \ \ 0]^T$.

Exercise 4

1a	1b	1c	2a	2b	2c	Exercise
3	**2**	**2**	**3**	**7**	**8**	**25 Points**

1. In 1798 Reverend T.R. Malthus postulated the following dynamical system to model growth of human population:
$$\dot{x}(t) = rx(t) \tag{2}$$
where $r > 0$, $x(t) \in \mathbb{R}$ and $t \in \mathbb{R}_+$.

   (a) Is the system (2) linear? Is it autonomous? What is its dimension?

   (b) Compute all equilibria of the system and determine their stability.

   (c) Compute the solution $x(t)$ of system (2) starting at $x(0) = x_0 > 0$. What is its limit as $t \to \infty$?

2. Subsequent research in population dynamics postulated the following system to model population growth in the presence of limited resources:
$$\dot{x}(t) = rx(t)\left(1 - \frac{x(t)}{K}\right) \tag{3}$$
Here $r > 0$, $K > 0$, $x(t) \in \mathbb{R}$ and $t \in \mathbb{R}_+$.

   (a) Is system (3) linear? Is it autonomous? What is its dimension?

   (b) Compute all equilibria of the system and determine their stability using linearization.

   (c) Show that
$$x(t) = \frac{Kx_0e^{rt}}{K + x_0\left(e^{rt} - 1\right)}$$
is the solution of system (3) starting at initial condition $x(0) = x_0 > 0$. Compute the limit of the solution as $t \to \infty$ and relate to your answer to part (2b). Comment on the relation of this solution and the one computed in part (1c) as $K \to \infty$.