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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4	Exercise
5	5	10	5	25 Points

Consider the system:

$$\dot{x}(t) = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u_2(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t) \qquad (1)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t). \qquad (2)$$

1. Is the system autonomous? Is it linear? How many states, inputs, and outputs does the system have? Is the system stable?
2. Compute the transfer function $G_2(s)$ from the input u_2 to the output y (assuming $u_1(t) = 0$ for all $t \geq 0$) and provide its poles and zeros.
3. You and a friend from EPFL are in charge of designing a state feedback controller for this system. Your job was to design the control law for the input $u_1(t)$, but before you could discuss the specifics of the task, your friend went ahead and designed a controller of the form $u_2(t) = K_2 x(t)$. To make matters worse, your friend has also forgotten what gain matrix K_2 he used. Luckily, you were able to perform a frequency analysis of the resulting closed loop SISO system and experimentally identify the transfer function $G(s)$ from u_1 to y as

$$G(s) = \frac{1}{s^2 - 1}.$$

   Reconstruct the time domain description of the closed loop SISO system using the controllable canonical form. Is the resulting time domain description unique? Is the system stable? Is it possible to design a feedback controller $u_1(t) = K_1 x(t)$ that stabilizes the system? Is it necessary to identify the gain matrix K_2 that your friend implemented to accomplish the last task?

4. Your friend from EPFL disagrees with the frequency analysis you performed and insists that the transfer function from u_1 to y for the closed loop SISO system is given by
$$G(s) = \frac{s+4}{s+2}.$$
Is this transfer function strictly proper? If possible, reconstruct the time domain description of this system. Support or argue against your friend's claim that this transfer function represents the true closed loop SISO system.

Exercise 2

1	2	3	4	Exercise
5	7	7	6	25 Points

Consider the system

$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -x_2(t) + \sin(x_1(t)) \\ -x_3(t) + a x_1(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} u(t), \qquad (3)$$
$$y(t) = x_2(t) + x_3(t),$$

where $x(t) = \begin{bmatrix} x_1(t) \; x_2(t) \; x_3(t) \end{bmatrix}^T \in \mathbb{R}^3$ is the state, $u(t) = \begin{bmatrix} u_1(t) \; u_2(t) \end{bmatrix}^T \in \mathbb{R}^2$ is the input, $y(t) \in \mathbb{R}$ is the output of the system, and $a \in \mathbb{R}$ is a constant parameter.

1. Determine all equilibrium points of (3) such that $0 \le x_1 < 2\pi$. Assume, for the time being, that $u(t) = 0$ for all $t \ge 0$.

2. Determine the stability of the equilibrium points of part 1, using linearization.

3. Linearize the system around the origin. Write the resulting linear system in state space form. For which values of α is the linearized system controllable? For which values of α is it observable?

4. Determine gains $k_1, k_2 \in \mathbb{R}$, and a function $g(x(t)) \in \mathbb{R}$, so that by applying the nonlinear control law
$$u_1(t) = g(x(t)) - k_1 x_1(t) - k_2 x_2(t),$$
$$u_2(t) = -k_1 x_1(t) - k_2 x_2(t),$$
to system (3), we obtain a linear system whose eigenvalues are all equal to -1.

Exercise 3

1	2	3	Exercise
7	10	8	25 Points

Consider a diagonalizable matrix $A \in \mathbb{R}^{n\times n}$ with real eigenvalues. The Backward Euler method generates a discrete time approximation x_k for $k = 0, 1, \ldots$ of

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0$$

by sampling every $\delta > 0$ seconds and setting

$$x_{k+1} = x_k + \delta A x_{k+1}.$$

1. Derive the dynamics of the discrete time system

   $$x_{k+1} = f\left(x_k\right)$$

   generated by the Backward Euler method. Is the discrete time system linear? Is it time invariant? Is it autonomous?
   (Hint: You may assume for the time being that the matrix $(I - \delta A) \in \mathbb{R}^{n\times n}$ is invertible.)

2. Compute the eigenvalues of the dynamics of the discrete time system in part 1 as a function of δ and the eigenvalues of the matrix A. Hence derive conditions on δ under which the hint in part 1 is indeed true.

3. Derive conditions on δ under which the discrete time system in part 1 is asymptotically stable. (You may again assume that the hint in part 1 holds.)

Exercise 4

1	2	3	4	Exercise
7	3	8	7	25 Points

A mass m is attached to a fixed point P by a spring of constant K and is allowed to move in the vertical plane under gravitational acceleration g. Assume $P \in \mathbb{R}^2$ is located at the origin and let $d = (x_1, x_3) \in \mathbb{R}^2$ denote the position of the mass and $v = (x_2, x_4) \in \mathbb{R}^2$ its velocity, as shown in Figure 1 . The spring applies force $-Kd$ for some $K > 0$ to the mass. The movement of the mass is also subject to an aerodynamic drag force of magnitude $-\alpha v$, for some $\alpha > 0$.

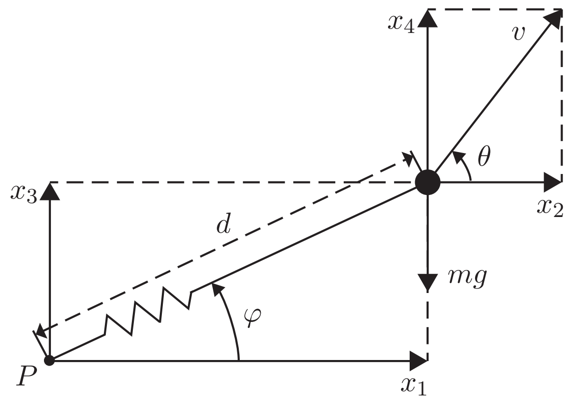

Figure 1: Mass on a spring

1. Derive the equations of motion of the system in state space form using x_1, x_2, x_3 and x_4 as states and x_1, x_3 as outputs.
   (Hint: $\cos\varphi = x_1/\sqrt{x_1^2 + x_3^2}$, $\cos\theta = x_2/\sqrt{x_2^2 + x_4^2}$)

2. Is the system linear? Is it time invariant? Is it autonomous?

3. Compute the equilibria of the system and determine their stability.
   (Hint: $\text{Det} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \text{Det}\,[A] \cdot \text{Det}\,[B]$)

4. For which states is the energy of the system

$$V(x) = \frac{1}{2}K\left(x_1^2 + x_3^2\right) + \frac{1}{2}m\left(x_2^2 + x_4^2\right) + mgx_3$$

   decreasing?