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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	Exercise
9	7	9	25 Points

1. According to the definition of estimation error $e_n = x_n - \hat{(x_n)}$, subtracting (**??**) from (**??**) yields to

$$e_{n+1} = x_{n+1} - \hat{x}_{n+1} = (A - LC)e_n \quad (1)$$

Regarding the error dynamics in (1), for every $x_0 \in \mathbb{R}^n$, $\hat{x_n} \to x_n$ as n tends to infinity iff $|\lambda_i(A - LC)| < 1$ where λ_i are eigenvalues of matrix $A - LC$.

2. The observability matrix is $\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 24 \end{bmatrix}$. Since $rank(\mathcal{O}) = 3$ then the system is **observable**.

The controllability matrix is $\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & -8 & 48 \\ 0 & -12 & 64 \\ 1 & -6 & 24 \end{bmatrix}$. Since $rank(\mathcal{C}) = 3$ then the system is **controllable** too.
When $u_n \equiv 0$, the system (**??**) is simplified as $x_{n+1} = Ax_n$; on the other hand, since $\lambda_i(A) = -2$ for $i = 1, 2, 3$ then the system is **unstable**.

3. Regarding the dimension matrixes C and A, $dim(L) = 3 \times 1$.
Now suppose $L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$; we are interested in observer gain, L, such that $\lambda_i(A-LC) = 0.5$ for $i = 1, 2, 3$.

$$A - LC = \begin{bmatrix} 0 & 0 & -8 - l_1 \\ 1 & 0 & -12 - l_2 \\ 0 & 1 & -6 - l_3 \end{bmatrix} \Rightarrow \lambda^3 + (6 + l_3)\lambda^2 + (12 + l_2)\lambda + (8 + l_1) = (\lambda - 0.5)^3 \quad (2)$$

Where λ is the eigenvalue of the matrix $A - LC$.
Computing the polynomial coefficients of (2) yields to $l_1 = -8.125$, $l_2 = -11.25$, and $l_3 = -7.5$.

Exercise 2

1	2	3	4	5	Exercise
3	5	6	7	4	25 Points

1. The dimension of the system is two (the highest order of derivatives). The system is not autonomous since the differential equation depends on the independent variable (t). The system is not linear due to the presence of $y(t)^3$.

2. Using the suggested state assignments the second order differential equation may be rewritten as

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_2 - x_1^3 + x_1 + \gamma cos(\omega t) \end{bmatrix} = f(x) \ . \tag{3}$$

3. To find the equilibria of the unforced system ($\gamma = 0$)we set $f(x) = 0$. We obtain the following set of algebraic equations:

$$0 = x_2$$
$$0 = -x_2 - x_1^3 + x_1$$

For all equilibria of the system $x_2^* = 0$ and $x_1(x_1^2 - 1) = 0$. This implies that there exist three equilibria: $\{(0,0),(1,0),(-1,0)\}$.

4. We linearize around $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$\frac{\partial f}{\partial x}(0,0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \tag{4}$$

We find the eigenvalues for this matrix: $\frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}+1}{2}$. One of them is positive, so this equilibrium is unstable.

We repeat the above process for the rest two equilibria,

$$\frac{\partial f}{\partial x}(1,0) = \frac{\partial f}{\partial x}(-1,0) = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \tag{5}$$

Both equilibria have eigenvalues $-0.5 \pm 0.5\sqrt{7}i$. The eigenvalues have negative real part, so both equilibria are locally stable.

5. The vector lines should diverge from the unstable equilibrium (0,0) and converge to both stable equilibria (±1,0).

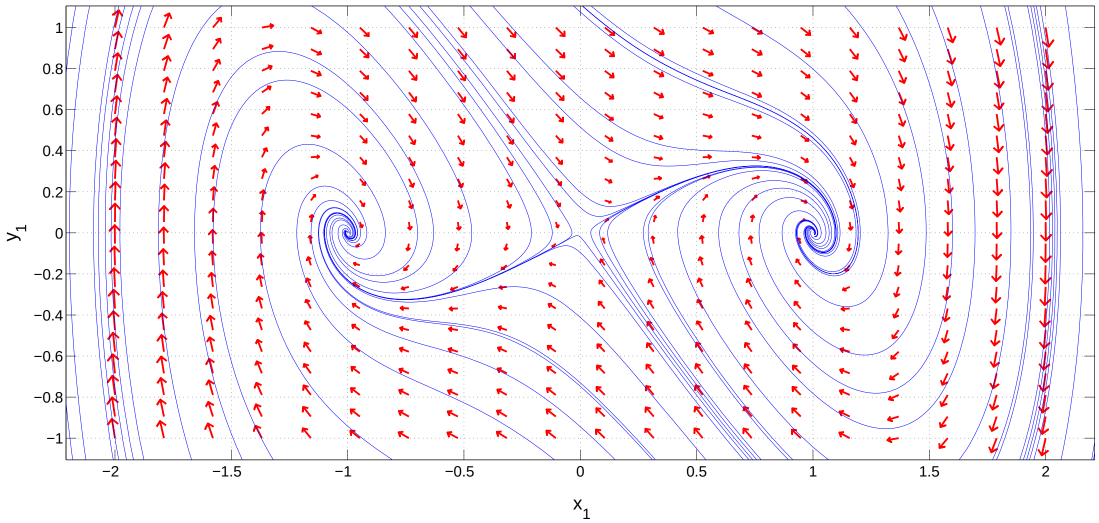

Figure 1: Phase Plane plot

Exercise 3

1	2	3	4	Exercise
5	7	6	7	25 Points

1. Given that $\hat{x} = Tx$ for the derivative of $\hat{x}$ we have: $\dot{\hat{x}} = T\dot{x}$. Replacing $\hat{x}$ and $\dot{\hat{x}}$ with the previous equations in (**??**), (**??**) we have:

$$T\dot{x} = \hat{A}Tx + \hat{B}u \Leftrightarrow \dot{x} = T^{-1}\hat{A}Tx + T^{-1}\hat{B}u \quad (6)$$
$$y = \hat{C}Tx \quad (7)$$

Comparing (6) and (7) with the initial system (**??**) and (**??**) we have:

$$\hat{A} = TAT^{-1} \quad (8)$$
$$\hat{B} = TB \quad (9)$$
$$\hat{C} = CT^{-1} \quad (10)$$

2. We start by computing the eigenvalues and the eigenvectors of matrix A. The eigenvalues of A are given by the characteristic equation:

$$
\begin{aligned}
\det(\lambda I - A) &= 0 \Leftrightarrow \\
\begin{vmatrix} \lambda + 2.5 & -0.5 \\ -0.5 & \lambda + 2.5 \end{vmatrix} &= 0 \Leftrightarrow \\
(\lambda + 2.5)^2 - 0.5^2 &= 0 \Leftrightarrow \\
(\lambda + 3)(\lambda + 2) &= 0 \Leftrightarrow \lambda_1 = -3 \text{ and } \lambda_2 = -2
\end{aligned}
$$

The eigenvector for λ_1 is:

$$
\begin{aligned}
Aw_1 &= \lambda_1 w_1 \Leftrightarrow \\
\begin{bmatrix} -2.5 & 0.5 \\ 0.5 & -2.5 \end{bmatrix} \begin{bmatrix} w_{11} \\ w_{12} \end{bmatrix} &= -3 \begin{bmatrix} w_{11} \\ w_{12} \end{bmatrix} \Leftrightarrow \\
\begin{bmatrix} -2.5w_{11} + 0.5w_{12} \\ 0.5w_{11} - 2.5w_{12} \end{bmatrix} &= \begin{bmatrix} -3w_{11} \\ -3w_{12} \end{bmatrix} \Leftrightarrow \\
\begin{bmatrix} 0.5w_{11} + 0.5w_{12} \\ 0.5w_{11} + 0.5w_{12} \end{bmatrix} &= 0 \Leftrightarrow w_{11} = -w_{12}
\end{aligned}
$$

The same way for the eigenvector for λ_2 can by found that $w_{21} = w_{22}$.

Therefore, we have found that the eigenvalues of A are distinct, its eigenvectors are linearly independent, which implies that matrix $W = [w_1 \; w_2]$ is invertible and that matrix A is diagonalizable. In such a case matrix A can be written as: $A = W \Lambda W^{-1}$, where:

$$
W = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}.
$$

It is easy to see now that if we select $T = W^{-1}$ and $\hat{A} = \Lambda$ equation (8) is satisfied. Concluding, the fact that matrix A is diagonalizable allows us to compute matrix T.

3. We start by computing the matrices $\hat{B}$ and $\hat{C}$, using (9) and (10).

$$
\begin{aligned}
\hat{B} &= TB \Leftrightarrow \\
&= W^{-1} B \Leftrightarrow \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Leftrightarrow \\
&= \begin{bmatrix} 0 \\ \frac{2}{\sqrt{2}} \end{bmatrix}
\end{aligned}
$$

and

$$\begin{aligned}
\hat{C} &= CT^{-1} \Leftrightarrow \\
&= CW \Leftrightarrow \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \Leftrightarrow \\
&= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}
\end{aligned}$$

Therefore, the change of coordinates leads in a Kalman decomposition, from which it is clear that one of the states is uncontrollable and that both states are observable. Thus, the system is observable but uncontrollable. Given that the second system is a Kalman decomposition of the initial system, the initial is system is observable and uncontrollable too.

4. The eigenvalues of the initial system w_1 and w_2 are linearly independent. All initial conditions can be written as a linear combination of w_1 and w_2. For an initial condition in the form $x_0 = \alpha_1 w_1 + \alpha_2 w_2$ the zero input response is $x(t) = \alpha_1 e^{\lambda_1 t} w_1 + \alpha_2 e^{\lambda_2 t} w_2$. Given that $d_1 = \lambda_1$, to have a zero input response in the given form $x(t) = x(0)e^{d_1 t}$ the initial condition should be $x(0) = w_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$.

Exercise 4

1	2	3	4	Exercise
7	6	5	7	25 Points

1. A) If the switch s is open then by obtaining the circuit we have that:

$$\begin{aligned} V_i(t) &= V_c(t) + V_{R_1}(t) \\ \Rightarrow V_i(t) &= V_c(t) + i_c(t)R_1 \\ \Rightarrow V_i(t) &= V_c(t) + CR_1\frac{dV_c(t)}{dt} \\ \Rightarrow \frac{dV_c(t)}{dt} &= -\frac{1}{CR_1}V_c(t) + \frac{1}{CR_1}V_i(t) \end{aligned} \tag{11}$$

$$\begin{aligned} V_L(t) &= V_{R_2}(t) \\ \Rightarrow L\frac{di_L(t)}{dt} &= -R_2 i_L(t) \\ \Rightarrow \frac{di_L(t)}{dt} &= -\frac{R_2}{L}i_L(t) \end{aligned} \tag{12}$$

The output $V_0(t)$ is

$$\begin{aligned} V_0(t) &= V_L(t) \\ \Rightarrow V_0(t) &= -R_2 i_L(t) \end{aligned} \tag{13}$$

Based on $(1) - (3)$ the state space equations of the system when the switch is open are:

$$\frac{d}{dt}\begin{bmatrix} V_c(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{CR_1} & 0 \\ 0 & -\frac{R_2}{L} \end{bmatrix}\begin{bmatrix} V_c(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{CR_1} \\ 0 \end{bmatrix} V_i(t) \tag{14}$$

$$V_0(t) = \begin{bmatrix} 0 & -R_2 \end{bmatrix}\begin{bmatrix} V_c(t) \\ i_L(t) \end{bmatrix} \tag{15}$$

Therefore we can define the following matrixes:

$$A = \begin{bmatrix} -\frac{1}{CR_1} & 0 \\ 0 & -\frac{R_2}{L} \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{CR_1} \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & -R_2 \end{bmatrix} \quad D = 0 \tag{16}$$

B) In the case where the switch s is closed, by applying the Kirchhoff law we have:

$$\begin{aligned} i_c(t) &= i_L(t) + i_R(t) \\ \Rightarrow C\frac{dV_c(t)}{dt} &= i_L(t) + \frac{V_i(t) - V_c(t)}{R} \\ \Rightarrow \frac{dV_c(t)}{dt} &= -\frac{1}{RC}V_c(t) + \frac{1}{C}i_L(t) + \frac{1}{RC}V_i(t) \end{aligned} \tag{17}$$

where $R = R_1//R_2 = \frac{R_1 R_2}{R_1+R_2}$

$$\begin{aligned} V_L(t) &= V_i(t) - V_c(t) \\ \Rightarrow L\frac{di_L(t)}{dt} &= V_i(t) - V_c(t) \\ \Rightarrow \frac{di_L(t)}{dt} &= -\frac{1}{L}V_c(t) + \frac{1}{L}V_i(t) \end{aligned} \tag{18}$$

The state space representation for this case is:

$$\frac{d}{dt}\begin{bmatrix} V_c(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} V_c(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{bmatrix} V_i(t) \tag{19}$$

$$V_0(t) = \begin{bmatrix} -1 & 0 \end{bmatrix}\begin{bmatrix} V_c(t) \\ i_L(t) \end{bmatrix} + V_i(t) \tag{20}$$

Therefore we can define the following matrixes:

$$A = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{bmatrix} \quad C = \begin{bmatrix} -1 & 0 \end{bmatrix} \quad D = 1 \tag{21}$$

2. A) In order to check if the system is controllable we can check if the controllability matrix P has full rank. For the first case where the switch is open P is

$$\begin{aligned} P &= \begin{bmatrix} B & AB \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{CR_1} & -\frac{1}{C^2R_1^2} \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{22}$$

Matrix P has not full rank, so the system is uncontrollable. Since the states are decoupled and the input affects only the first of the states, we should expect also by intuition that the system would be uncontrollable.
B) For the case where the switch is closed the matrix P is:

$$\begin{aligned} P &= \begin{bmatrix} B & AB \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{RC} & -\frac{1}{R^2C^2} + \frac{1}{LC} \\ \frac{1}{L} & -\frac{1}{RLC} \end{bmatrix} \end{aligned} \tag{23}$$

Matrix P has full rank, so the system is controllable.

3. A) In order to check if the system is observable we can check if the observability matrix Q has full rank. For the first case Q is

$$\begin{aligned} Q &= \begin{bmatrix} C \\ CA \end{bmatrix} \\ &= \begin{bmatrix} 0 & -R_2 \\ 0 & \frac{R_2^2}{L} \end{bmatrix} \end{aligned} \tag{24}$$

Matrix Q has not full rank, so the system is unobservable.
B) If the switch is closed the matrix Q is:

$$\begin{aligned} Q &= \begin{bmatrix} C \\ CA \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ \frac{1}{RC} & -\frac{1}{C} \end{bmatrix} \end{aligned} \tag{25}$$

Since Q has full rank the system is observable.

4. A) $0 \leq t < 1$:
For this case the switch is open and the system is described by equations (4) and (5). Since $V_i(t) = 0 \;\; \forall t \geq 0$, the response of the system for this interval is the zero input response. If $x(t) = \begin{bmatrix} V_c(t) & i_L(t) \end{bmatrix}^T$ then

$$x(t) = \Phi(t)x_0 = e^{At}x_0 \tag{26}$$

where $\Phi(t)$ is the state transition matrix and $x_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ is the initial condition. In order to compute $\Phi(t)$ the eigenvalues and the eigenvector matrix need to be computed. By obtaining A from equation (6), the eigenvalues are:

$$\begin{aligned} \lambda_1 &= -\frac{1}{CR_1} \quad \lambda_2 = -\frac{R_2}{L} \\ \Rightarrow \lambda_1 &= -\frac{3}{2} \quad \lambda_2 = -\frac{4}{3} \end{aligned} \tag{27}$$

But

$$Aw_i = \lambda_i w_i \text{ for } i = 1, 2 \tag{28}$$

Where w_i is the eigenvector that corresponds to eigenvalue λ_i. Based on the last equation we can find that

$$w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{29}$$

So the eigenvector matrix is

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{30}$$

Since $\Phi(t) = e^{At} = We^{\Lambda t}W^{-}1$ where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{31}$$

By using (20) and (21)

$$\Phi(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e^{\Lambda t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{-\frac{1}{CR_1}t} & 0 \\ 0 & e^{-\frac{R_2}{L}t} \end{bmatrix} \tag{32}$$

So from (16):

$$x(t) = \begin{bmatrix} V_c(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} e^{-\frac{1}{CR_1}t} & 0 \\ 0 & e^{-\frac{R_2}{L}t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-\frac{1}{CR_1}t} \\ e^{-\frac{R_2}{L}t} \end{bmatrix} \quad (33)$$

$$y(t) = V_0(t) = -R_2 i_L(t) = -R_2 e^{-\frac{R_2}{L}t} \quad (34)$$

By substituting the numerical values we have:

$$x(t) = \begin{bmatrix} e^{-\frac{3}{2}t} \\ e^{-\frac{4}{3}t} \end{bmatrix} \quad (35)$$

$$y(t) = -\frac{2}{3} e^{-\frac{4}{3}t} \quad (36)$$

B) $t \geq 1$:
At time $t = 1$ the switch closes and remains closed for the rest of the time. This time the matrixes A, B, C and D are given by (11). The characteristic equation is $\lambda^2 + \frac{1}{RC}\lambda + \frac{1}{LC} = 0$, so the eigenvalues are of the form:

$$\lambda_i = \frac{-\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{4}{LC}}}{2} \text{ for } i = 1, 2 \quad (37)$$

Hence, by using the numerical values, the eigenvalues are:

$$\lambda_1 = -1 \quad \lambda_2 = -2 \quad (38)$$

But since

$$Aw_i = \lambda_i w_i \text{ for } i = 1, 2 \quad (39)$$

Where w_i is the eigenvector that corresponds to eigenvalue λ_i we can find that

$$w_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } w_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (40)$$

So the eigenvector matrix is

$$W = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad (41)$$

This time the transition matrix $\Phi(t)$ would be:

$$\Phi(t) = W e^{\Lambda t} W^{-1} \quad (42)$$

with

$$W^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

So

$$\Phi(t) = \begin{bmatrix} -e^{-t} + 2e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix} \quad (43)$$

Therefore, the system's response for $t \geq 1$ is

$$\begin{aligned} x(t) &= \Phi(t-1)x(1) = \begin{bmatrix} 0.11e^{-t} + 1.35e^{-2t} \\ 0.22e^{-t} + 1.35e^{-2t} \end{bmatrix} \\ y(t) &= -0.11e^{-t} - 1.35e^{-2t} \end{aligned} \tag{44}$$

where

$$x(1) = \begin{bmatrix} e^{-\frac{3}{2}} \\ e^{-\frac{4}{3}} \end{bmatrix}$$

So based on (26) and (34) the response of the system is

$$y(t) = \begin{cases} -\frac{2}{3}e^{-\frac{4}{3}t} & \text{if } 0 \leq t < 1 \\ -0.11e^{-t} - 1.35e^{-2t} & \text{if } t \geq 1 \end{cases} \tag{45}$$