Automatic Control Laboratory
ETH Zurich
Prof. J. Lygeros

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Signal and System Theory II
4. Semester, BSc

Solutions

1 Exercise 1

1	2	3	4	Aufgabe
5	5	8	7	25 Punkte

1. To find the equilibria of the system $\dot{x} = Ax$ we set $\dot{x} = 0$. So $Ax = 0$:

$$\begin{aligned} Ax = 0 \quad &\Leftrightarrow \\ \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad &\Leftrightarrow \\ x_1 = x_2 \end{aligned}$$

Figure 1 shows the locations of the equilibria of the system in the x_1 - x_2 plane.

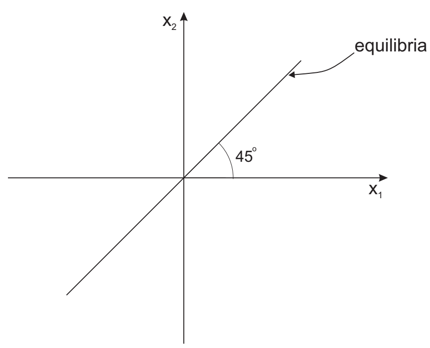

Figure 1: Equilibria locations in x_1 - x_2 plane.

2. Stability of $\hat{x} = 0$ can be determined from the eigenvalues of the matrix A, i.e. the roots of the characteristic polynomial $|\lambda I - A|$

$$\begin{aligned} |\lambda I - A| &= 0 \Rightarrow \\ \begin{vmatrix} \lambda + \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda + \frac{1}{2} \end{vmatrix} &= 0 \Rightarrow \\ \lambda^2 + \lambda + \frac{1}{4} - \frac{1}{4} &= 0 \Rightarrow \\ \lambda(\lambda + 1) &= 0 \Rightarrow \end{aligned}$$

$$\lambda = 0 \ , \lambda = -1 \tag{1}$$

The eigenvalues of matrix A are distinct and non-positive so the system is stable. Since $\lambda = 0$ us an eigenvalue, the system is not asymptotically stable.

3. The eigenvectors of the matrix A can be computed by the equation: $Aw_i = \lambda_i w_i$ where λ_i is eigenvalue of the matrtix A.

For $\lambda = 0$:

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -\frac{1}{2}x_1 + \frac{1}{2}x_2 \\ \frac{1}{2}x_1 - \frac{1}{2}x_2 \end{bmatrix} = 0 \Rightarrow x_1 = x_2$$

For $\lambda = -1$:

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} -\frac{1}{2}x_1 + \frac{1}{2}x_2 \\ \frac{1}{2}x_1 - \frac{1}{2}x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} \Rightarrow x_2 = -x_1$$

So the two eigenvectors are:

$$w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad w_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and are shown in Fig. 2.

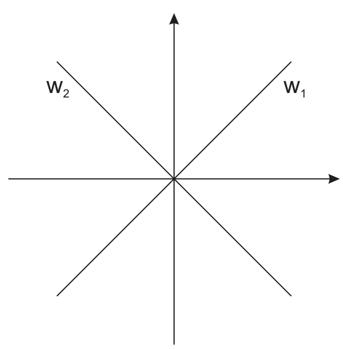

Figure 2: System eigenvectors.

The state transition matrix e^{At} can be computed by the equation:

$$e^{At} = W e^{\Lambda t} W^{-1} \tag{2}$$

where

$$W = \begin{bmatrix} w_1 & w_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

Replacing W and Λ in Eq. 2 we have:

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & e^{-t} \\ 1 & -e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} + \frac{e^{-t}}{2} & \frac{1}{2} - \frac{e^{-t}}{2} \\ \frac{1}{2} - \frac{e^{-t}}{2} & \frac{1}{2} + \frac{e^{-t}}{2} \end{bmatrix} \end{aligned}$$

4. Given the transition matrix e^{At} and an initial condition $x_0 = [x_{01}\ x_{02}]^T$ we have:

$$\begin{aligned} x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= e^{At} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} + \frac{e^{-t}}{2} & \frac{1}{2} - \frac{e^{-t}}{2} \\ \frac{1}{2} - \frac{e^{-t}}{2} & \frac{1}{2} + \frac{e^{-t}}{2} \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \\ &= \begin{bmatrix} \frac{x_{01}}{2} + \frac{e^{-t}x_{01}}{2} + \frac{x_{02}}{2} - \frac{e^{-t}x_{02}}{2} \\ \frac{x_{01}}{2} - \frac{e^{-t}x_{01}}{2} + \frac{x_{02}}{2} + \frac{e^{-t}x_{02}}{2} \end{bmatrix} \end{aligned}$$

When $t \to \infty$:

$$\lim_{t\to\infty} x(t) = \begin{bmatrix} \frac{x_{01}+x_{02}}{2} \\ \frac{x_{01}+x_{02}}{2} \end{bmatrix}$$

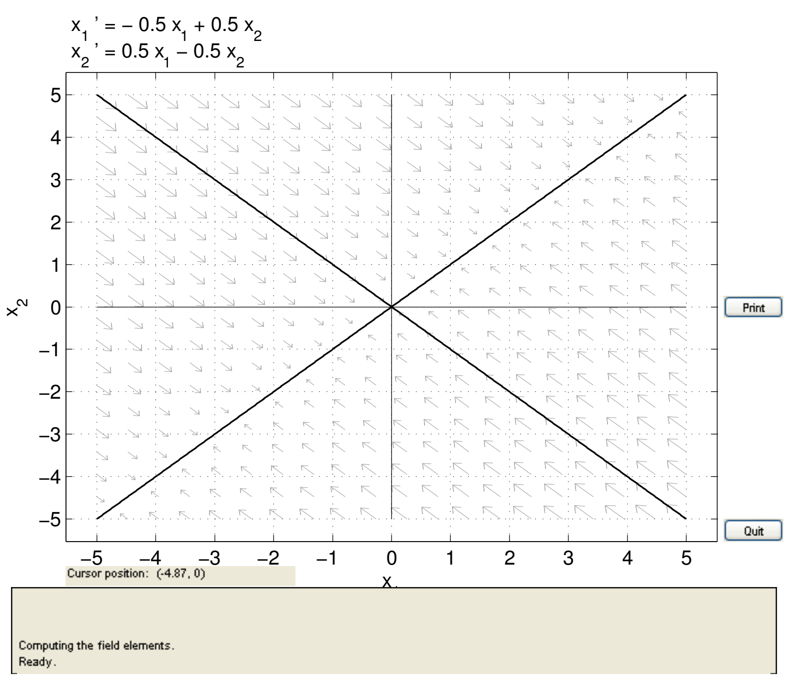

Figure 3: System phase plane.

2 Exercise 2

1	2	3	4	Exercise
5	5	5	10	25 Points

1. The system is uncontrollable since there is no input!

   To check the observability, we compute the observability matrix

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

   Q has full rank, so the system is observable.

2. First, note the basic results $x_1 = Ax_0$ and $x_2 = Ax_1 = A^2x_0$. Therefore

$$\begin{aligned} y_0 = Cx_0 &= \begin{bmatrix} 100 \end{bmatrix} x_0 && (4)\\ y_1 = Cx_1 = CAx_0 &= \begin{bmatrix} 010 \end{bmatrix} x_0 && (5)\\ y_2 = Cx_2 = CA^2x_0 &= \begin{bmatrix} 001 \end{bmatrix} x_0 && (6) \end{aligned}$$

   Stacking y_0, y_1, and y_2, we obtain the relation

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_0 = x_0. \tag{7}$$

3. By definition,

$$\begin{aligned} e_k = x_k - \hat{x}_k \Rightarrow e_{k+1} &= x_{k+1} - \hat{x}_{k+1} && (8)\\ &= Ax_k - A\hat{x}_k - L(y_k - C\hat{x}_k) && (9)\\ &= A(x_k - \hat{x}_k) - LC(x_k - \hat{x}_k) && (10)\\ &= (A - LC)e_k && (11) \end{aligned}$$

$$\begin{aligned} A - LC &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} && (12)\\ &= \begin{bmatrix} -l_1 & 1 & 0 \\ -l_2 & 0 & 1 \\ -l_3 & 0 & 0 \end{bmatrix} && (13) \end{aligned}$$

4. Our system is discrete time, thus we should choose $\lambda = \frac{1}{2}$ for the dynamics of the error to be stable. Therefore, we would like to choose L such that the matrix has three eigenvalues at $\lambda = \frac{1}{2}$. Next, we solve the characteristic equation of $A - LC$

$$\begin{aligned} det(\lambda - (A - LC)) &= det \begin{bmatrix} \lambda + l_1 & -1 & 0 \\ l_2 & \lambda & -1 \\ l_3 & 0 & \lambda \end{bmatrix} && (14)\\ &= \lambda(\lambda^2 + l_1\lambda + l_2) + l_3 && (15)\\ &= \lambda^3 + l_1\lambda^2 + l_2\lambda + l_3 && (16) \end{aligned}$$

We thus have our characteristic polynomial in terms of the variables of L. However, we require the eigenvalues to all have value $\lambda = \frac{1}{2}$. Thus, we require

$$\begin{aligned} det(\lambda - (A - LC)) &= (\lambda - \frac{1}{2})^3 && (17) \\ &= \lambda^3 - \frac{3}{2}\lambda^2 + \frac{3}{4}\lambda - \frac{1}{8} && (18) \end{aligned}$$

Which, when compared to the analytical characteristic polynomial, leads us to choose our gain matrix L to be

$$L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{4} \\ -\frac{1}{8} \end{bmatrix}. \qquad (19)$$

In this part, we designed a stable observer with eigenvalues at $\lambda = \frac{1}{2}$. Thus, as $k \to \infty$, $e_k \to 0$. Therefore, even if our initial state estimate $\hat{x}_0$ is inaccurate or we have noisy measurements y_k, our error is always guaranteed to remain stable, and generally head towards zero. This is not the case for the observer in part 2. However, given that the eigenvalues of the error are non-zero, if we have a non-zero initial error, our observer error in part 4 will never go to exactly zero (it will only converge towards zero) even with perfect measurement values. Under these same conditions, our observer in part 2 works perfectly since it does not depend on an initial guess of $\hat{x}_0$.

(Alternatively)

Part 2: Finite time convergence but prone to measurement noise.

Part 4: Requires infinite time to converge but more robust to noise (it takes into account all measurements, not just the first three).

3 Exercise 3

1	2	3	4	Aufgabe
5	8	7	5	25 Punkte

1. $P = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$: the system is controllable.

   $Q = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$: the system is observable.

   $$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 2 \end{vmatrix} = 0 \iff (\lambda - 1)(\lambda - 2) = 0 \implies \lambda_1 = 1 \text{ and } \lambda_2 = 2:$$

   the system is unstable.

2. $$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \implies$$

   $$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 1 \\ k_1 & 2 + k_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}$$

   $$\begin{aligned} G(s) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s - 1 & -1 \\ -k_1 & s - 2 - k_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{(s-1)(s-2-k_2) - k_1} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s - 2 - k_2 & 1 \\ k_1 & s - 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{s^2 - s - 2s + 2 - k_2 s + k_2 - k_1} \begin{bmatrix} s - 2 - k_2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{s^2 - (k_2 + 3)s + 2 + k_2 - k_1} \end{aligned}$$

3. We want the polynomials $(s+1)^2 = s^2 + 2s + 1$ and $s^2 - (k_2 + 3)s + 2 + k_2 - k_1$ to be equal, i.e.

   $$\begin{aligned} -(k_2 + 3) &= 2 \implies k_2 = -5 \\ 2 + k_2 - k_1 &= 1 \implies k_1 = -4 \end{aligned}$$

4. One can build an observer (since the system is observable). Alternatively, we can also differentiate y, since $x_2 = \dot{x}_1 - x_1 = \dot{y} - y$.

4 Exercise 4

1	2	3	Aufgabe
10	8	7	25 Punkte

1. We will use the voltages across the two capacitors of the circuit as our states, V_{C_1} and V_{C_2}. Using Kirchhoff's circuit laws, we have, for the currents:

$$I_1 = I_k + I_{C_1} \quad \text{(Op-Amp ideal)} \tag{20}$$

$$I_3 = I_{C_2} \quad \text{(Op-Amp ideal)} \tag{21}$$

For the voltages:

$$V_{C_1} - I_k K R_0 = 0 \Rightarrow I_k = \frac{V_{C_1}}{KR_0} \tag{22}$$

and

$$V_i - I_1 R_1 - I_3 R_3 = 0 \quad \text{(Op-Amp ideal)} \tag{23}$$

$$V_{C_1} + I_{C_2} R_2 + V_{C_2} = 0 \quad \text{(Op-Amp ideal)} \tag{24}$$

Furthermore, for the currents that pass through the two capacitors we have:

$$I_{C_1} = C_1 \dot{V}_{C_1} \tag{25}$$

$$I_{C_2} = C_2 \dot{V}_{C_2} \tag{26}$$

Using the above equations we can derive the following:

$$(23), (25), (26), (21) \Rightarrow V_i - \frac{R_1}{KR_0} V_{C_1} - C_1 R_1 \dot{V}_{C_1} - C_2 R_3 \dot{V}_{C_2} = 0 \tag{27}$$

From the equations:

$$(24), (26) \Rightarrow \dot{V}_{C_2} = -\frac{1}{C_2 R_2} V_{C_1} - \frac{1}{C_2 R_2} V_{C_2} \tag{28}$$

Substituting (28) to (27) we get:

$$\dot{V}_{C_1} = -\frac{1}{C_1 K R_0} V_{C_1} + \frac{R_3}{C_1 R_1 R_2} V_{C_1} + \frac{R_3}{C_1 R_1 R_2} V_{C_2} + \frac{1}{C_1 R_1} V_i \tag{29}$$

The output voltage V_0 is:

$$\begin{aligned} V_0 &= I_{C_2}R_2 + V_{C_2} + I_3R_3 && (30) \\ &= (R_3+R_2)C_2\dot{V}_{C_2} + V_{C_2} && (31) \\ &= (R_3+R_2)C_2(-\frac{1}{C_2R_2}V_{C_1} - \frac{1}{C_2R_2}VC_2) + V_{C_2} && (32) \\ &= -\frac{R_3}{R_2}V_{C_1} - V_{C_1} - \frac{R_3}{R_2}V_{C_2} && (33) \end{aligned}$$

Now we can derive the state space equations:

$$\begin{bmatrix} \dot{V}_{C_1} \\ \dot{V}_{C_2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1KR_0} + \frac{R_3}{C_1R_1R_2} & \frac{R_3}{C_1R_1R_2} \\ -\frac{1}{C_2R_2} & -\frac{1}{C_2R_2} \end{bmatrix} \begin{bmatrix} V_{C_1} \\ V_{C_2} \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1R_1} \\ 0 \end{bmatrix} V_i$$

and

$$V_0 = \begin{bmatrix} -\frac{R_3}{R_2} - 1 & -\frac{R_3}{R_2} \end{bmatrix} \begin{bmatrix} V_{C_1} \\ V_{C_2} \end{bmatrix}$$

So,

$$A = \begin{bmatrix} -\frac{1}{K}+1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & -1 \end{bmatrix}$$

$$D = 0$$

2. The transfer function of the system is computed by the formula:

$$G(s) = C(sI - A)^{-1}B + D$$

$$G = -\frac{2Ks + K}{(Ks^2 + s + 1)} \quad (34)$$

3. The A matrix becomes:

$$A = \begin{bmatrix} -\frac{1}{K}+1 & 1 \\ -1 & -1 \end{bmatrix} \quad (35)$$

The poles of the transfer function are:

$$s_2 = \frac{1}{2K}(-1 + \sqrt{1-4K})$$

$$s_2 = \frac{1}{2K}(-1 - \sqrt{1-4K})$$

The system is asymptotically stable for all $K > 0$. For $K \leq 0.25$ the state decays to 0 exponentially. For $K > 0.25$ the state oscillates like a sine-wave with exponentially decaying amplitude.