Automatic Control Laboratory — D-ITET
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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	5	Exercise
8	5	4	4	4	25 Points

1. We first note that the current through R_1 is equal to that through C_2, and given as follows:
$$i_{R_1} = i_{C_2} = \frac{V_{\text{in}} - v_{C_1} - v_{C_2}}{R_1}.$$
The current through C_1, on the other hand, is the above with the current through the other two resistors subtracted. We first replace R_2 and R_3 with their parallel equivalent circuit to simplify things:
$$\bar{R} := R_2 \parallel R_3 = \frac{R_2 R_3}{R_2 + R_3}$$
We can then write:
$$i_{C_1} = i_{R_1} - \frac{v_{C_1}}{\bar{R}}.$$
Using the relationship of current and voltage for capacitors C,
$$\frac{d}{dt} v_C = \frac{1}{C} i_C,$$
we can now get the above equations into standard form:
$$\begin{aligned} \frac{d}{dt} v_{C_1} &= \frac{1}{C_1} i_{C_1} = \frac{1}{C_1}\left(\frac{V_{\text{in}}}{R_1} - \frac{v_{C_1}}{R_1} - \frac{v_{C_2}}{R_1} - \frac{v_{C_1}}{\bar{R}}\right) \\ \frac{d}{dt} v_{C_2} &= \frac{1}{C_2}\left(\frac{V_{\text{in}}}{R_1} - \frac{v_{C_1}}{R_1} - \frac{v_{C_2}}{R_1}\right) \end{aligned} \tag{1}$$
we can now write the system in state-space form:
$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -\frac{1}{R_1 C_1} - \frac{1}{\bar{R} C_1} & -\frac{1}{R_1 C_1} \\ -\frac{1}{R_1 C_2} & -\frac{1}{R_1 C_2} \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{R_1 C_1} \\ \frac{1}{R_1 C_2} \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + 0u(t). \end{aligned} \tag{2}$$
The first entry of A can also be written as follows:
$$-\frac{1}{R_1 C_1} - \frac{1}{\bar{R} C_1} = -\frac{1}{R_1 C_1} - \frac{R_2 + R_3}{R_2 R_3 C_1} = -\frac{R_2 R_3 + R_1 R_2 + R_1 R_3}{R_1 R_2 R_3 C_1} \tag{3}$$

2. The system is asymptotically stable since it is passive and has dissipative components in it (the resistors). If the system were started at any state and left as it is with no input ($V_{\text{in}} = 0$), the states would decay to zero. Entering the given values, we get $\bar{R} = 1$ and the equations become:
$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + 0u(t). \end{aligned} \tag{4}$$

We can now write down the characteristic polynomial as

$$\det(\lambda I - A) = (\lambda + 2)(\lambda + 1) - 1 = \lambda^2 + 3\lambda + 1 \overset{!}{=} 0$$

which means the eigenvalues are

$$\lambda_{1,2} = \frac{-3 \pm \sqrt{9-4}}{2}$$

and hence both have strictly negative real parts. This means the system is stable as well as asymptotically stable.

3. Since the system is stable, we can for example use the energy in the capacitors as a Lyapunov function:
$$V(x) = \frac{1}{2}v_{C_1}^2 + \frac{1}{2}v_{C_2}^2 = \frac{1}{2}x^T I x^T$$
It now follows that:
$$A^T Q + QA = A^T I + IA = A + A^T$$
From this we get that $R = -(A + A^T) = -2A$, which is positive definite since A has negative eigenvalues and is symmetric. Since Q is positive definite and unique for this R, all conditions for asymptotic stability are satisfied.

4. We set $f(x) = Ax$ and check the properties:
   - $V(0) = 0$ holds trivially
   - $V(x \neq 0) \geq 0$ holds because we sum up squares of real numbers
   - The time derivative property is verified as follows:
   $$\frac{d}{dt}V(t) = \frac{1}{2}x^T I\dot{x} + \frac{1}{2}\dot{x}^T I x = \frac{1}{2}(x^T IAx + x^T A^T Ix) = \frac{1}{2}x^T(2A)x$$
   Since we know A has negative eigenvalues, $2A$ will also have negative eigenvalues and hence the derivative is ≤ 0.

5. If R_2 became zero, that would mean also $\bar{R}$ would become zero and the equations from part 1 would no longer hold. However, since there is still dissipative action in the system it would still be asymptotically stable. The output voltage would be 0 for all times, but $v_{C_2}(t)$ would follow the input in a low-pass fashion. We would not be able to distinguish different states of v_{C_2} by looking at the output, hence it would become unobservable.

Exercise 2

1	2	3	4	5	Exercise
6	4	4	8	3	25 Points

1. The controllability matrix is given by

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 2 & \frac{2}{100} - 2 \\ 1 & -1 & 1 \\ -1 & 2 & -3 \end{bmatrix}$$

   In order to determine whether the system is controllable, one has to determine whether it is full rank. This can either be done by computing all eigenvalues of P to see whether they are all nonzero or by computing the determinant and checking whether it is nonzero. Full rank means controllability, rank deficiency means an uncontrollable system.

   The determinant of the matrix here is

$$\det(P) = -2(-3+1) + (\frac{2}{100} - 2)(2-1) = 2 + \frac{2}{100} \neq 0$$

   and hence the system is controllable.

2. Since the second state does not depend on the other states, we would lose controllability. This can also be seen mathematically by writing down the modified controllability matrix:

$$\tilde{P} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

   which has only rank 1.

3. The eigenvalues of the original system are computed as follows:

$$\det(\lambda I - A) = \left(\lambda - \frac{1}{100}\right)(\lambda+1)(\lambda+1)$$

   which means the eigenvalues are $\frac{1}{100}, -1, -1$ and the system is unstable. It is neither stable, nor asymptotically stable.

4. The first simplified system only has the eigenvalues $-1, -1$ and is hence stable. For modeling purposes, these systems hence are greatly different and cannot be simplified like that.

   The second system retains the positive eigenvalue and even becomes diagonal, but upon closer inspection, it is neither controllable nor observable while the original system is. It is hence also a bad model.

5. No, this could not have happened, since invertible transforms conserve all the information in the state and hence do not change any of the stability, observability or controllability properties.

Exercise 3

1	2	3	4	5	Exercise
3	4	5	5	8	25 Points

1. No, the system is not linear. Yes, the system is autonomous. Yes, the system is time-invariant.

2. The unique equilibrium is $(0, 0)$. Linearization at the equilibrium gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

   The eigenvalues of the linearized system are -1 with multiplicity 2. Hence, the system is locally asymptotically stable.

3. The Lie derivative is given by

$$\nabla V_1(x_1, x_2)^\top f(x_1, x_2) = -2x_1^2 + 2x_1^2 x_2 - 2x_2^2$$

   Let $\epsilon > 0$ and $||(x_1, x_2)|| < \epsilon$. Then $|x_2| \leq \epsilon$. Using this fact in the above expression, we get

$$\nabla V_1(x_1, x_2)^\top f(x_1, x_2) \leq -2x_1^2 + 2\epsilon x_1^2 - 2x_2^2.$$

   Now pick $\epsilon \in (0, 1)$. Then, $\nabla V_1(x_1, x_2)^\top f(x_1, x_2) \leq -2(1-\epsilon)x_1^2 - 2x_2^2 < 0$ whenever $(x_1, y) \neq (0, 0)$. This leads again to the conclusion using Lyapunov's second method that the system is locally asymptotically stable.

4. Note that the Lie derivative is

$$\nabla V_1(x_1, x_2)^\top f(x_1, x_2) = -2x_1^2 + 2x_1^2 x_2 - 2x_2^2.$$

   Pick $(x_1, x_2) = (3, 3)$. The Lie derivative at this point positive (the value is 18). No, this does not mean that the system is not globally asymptotically stable.

5. The function is $V_2(x_1, x_2) = \ln(1 + x_1^2) + x_2^2$.

   (a) Note that $1 + x_1^2 \geq 1$. Using the fact that $\ln(\cdot)$ is a non-decreasing function we get $\ln(1 + x_1^2) \geq \ln(1) = 0$. Further $x_2^2 \geq 0$. Hence $V_2(x_1, x_2) \geq 0$. Using these facts, $V_2(x_1, x_2) = 0$ implies that $(x_1, x_2) = (0, 0)$ which is the equilibrium.

   (b) Computing the Lie derivative

$$\begin{aligned} \nabla V_2(x_1, x_2)^\top f(x_1, x_2) &= \frac{2x_1}{1 + x_1^2}(-x_1 + x_1 x_2) + 2x_2(-x_2) \\ &= \frac{-2x_1^2 + 2x_1^2 x_2 - 2x_2^2 - 2x_1^2 x_2^2}{1 + x_1^2} \\ &= \frac{-x_1^2 - x_1^2 x_2^2 - 2x_2^2 - (x_1 - x_1 x_2)^2}{1 + x_1^2} < 0. \end{aligned}$$

   From the above expression, we see that the Lie derivative is zero at the equilibirum $(0, 0)$.

   (c) We conclude that the system is globally asymptotically stable.

Exercise 4

1	2	3	4	Exercise
4	7	7	7	25 Points

1. The transfer function can be computed from the state space representation as follows

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+d & k \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{(s+d)s+k} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & -k \\ 1 & s+d \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{s^2+ds+k} \end{aligned}$$

   According to the Hurwitz criterion, all roots of a second-degree polynomial $P(s) = s^2 + a_1 s + a_0$ have negative real part if and only if $a_1 > 0$ and $a_0 > 0$. Therefore, $G(s)$ is stable if and only if $k > 0$ and $d > 0$.

2. Based on the content of the course, there are two possible solutions.

   Solution 1:

   Because the numerator is constant we can minimize the square of the magnitude of the denominator $D(\omega) := k - \omega^2 + dj\omega$ instead of maximizing the magnitude of the transfer function:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\omega}|D(\omega)|^2 &= \frac{\mathrm{d}}{\mathrm{d}\omega}\sqrt{(k-\omega^2)^2 + (d\omega)^2}^2 \\ &= \frac{\mathrm{d}}{\mathrm{d}\omega}(k-\omega^2)^2 + (d\omega)^2. \end{aligned}$$

   Next, using the chain rule one obtains

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\omega}(k-\omega^2)^2 + (d\omega)^2 &= -4\omega(k-\omega^2) + 2d^2\omega \\ &= -4\Big(\omega(k - \omega^2 - \tfrac{1}{2}d^2)\Big) \\ &= 4\omega\Big(\omega^2 - (k - \tfrac{1}{2}d^2)\Big). \end{aligned}$$

   Solving $4\omega\big(\omega^2 - (k - \frac{1}{2}d^2)\big) = 0$ one obtains the two critical points $\omega_1 = 0$ and $\omega_2 = \sqrt{k - \frac{1}{2}d^2}$. Taking the second derivative of the squared magnitude of the denominator results in

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}^2\omega}|D(\omega)|^2 &= \frac{\mathrm{d}}{\mathrm{d}\omega}4\omega\Big(\omega^2 - (k - \tfrac{1}{2}d^2)\Big) \\ &= 12\omega^2 - 4(k - \tfrac{1}{2}d^2) \end{aligned}$$

and it follows that $\frac{\mathrm{d}^2|D(\omega)|^2}{\mathrm{d}^2\omega}\Big|_{\omega=\omega_1} < 0$ and $\frac{\mathrm{d}^2|D(\omega)|^2}{\mathrm{d}^2\omega}\Big|_{\omega=\omega_2} > 0$. In other words, ω_2 minimizes $|D(\omega)|^2$ and it follows that the maximum of $|G(j\omega)|$ is at $\omega = \sqrt{k - \frac{1}{2}d^2}$.

Solution 2:
Comparing coefficients we obtain

$$\frac{1}{s^2 + ds + k} = \frac{K'\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5}$$

where $K' = k^{-1}$, $\zeta = \frac{1}{2}d\frac{1}{\sqrt{k}}$, and $\omega_n = \sqrt{k}$. Considering $d^2 < 2k$ it follows that $\zeta < \frac{1}{2}\sqrt{2k}\frac{1}{\sqrt{k}} = \frac{1}{\sqrt{2}}$. Therefore, the maximum magnitude occurs at $\omega = \omega_n\sqrt{1-\zeta^2} = \sqrt{k}\sqrt{1 - \frac{1}{2}\frac{d^2}{k}} = \sqrt{k - \frac{1}{2}d^2}$.

3. We compute the magnitude of the transfer functions $G_i(j\omega)$ at $\omega = 2$:

$$|G_1(2j)| = \frac{|-12j|}{|-4 + 12j + 4|} = 1, \tag{6}$$

$$|G_2(2j)| = \frac{|0|}{|(2j+1)^2|} = 0, \tag{7}$$

$$|G_3(2j)| = \frac{|1|}{|-1 + 1/2j + 1|} = 2. \tag{8}$$

Comparing these magnitudes to the amplitude of the zero-state responses depicted in Figure 1 it can be seen that the zero-state response (d) is consistent with $G_1(s)$, the zero-state response (b) is consistent with $G_2(s)$, and the zero-state response (c) is consistent with $G_3(s)$.

4. The transfer function $G_{\Sigma_1}(s)$ has an unstable pole at $s = 1$. Therefore, the variable P in the lecture notes of the Nyquist stability criterion is one. This implies that for stability of the closed loop system we need to satisfy the equation $N = -P = -1$, where N denotes the number of clockwise encirclements of the Nyquist curve around the critical point $-\frac{1}{K}$. Therefore we require one counter-clockwise encirclement of the point $-\frac{1}{K}$ to ensure stability of the closed loop. By considering the Nyquist diagram shown in Figure 2 this is ensured when $-\frac{1}{K} < \Re\{p\}$, i.e, $-\frac{1}{K} < -\frac{1}{108}$, and $-\frac{1}{K} > -1/20$ hold. In other words, the closed-loop is stable for $20 < K < 108$.

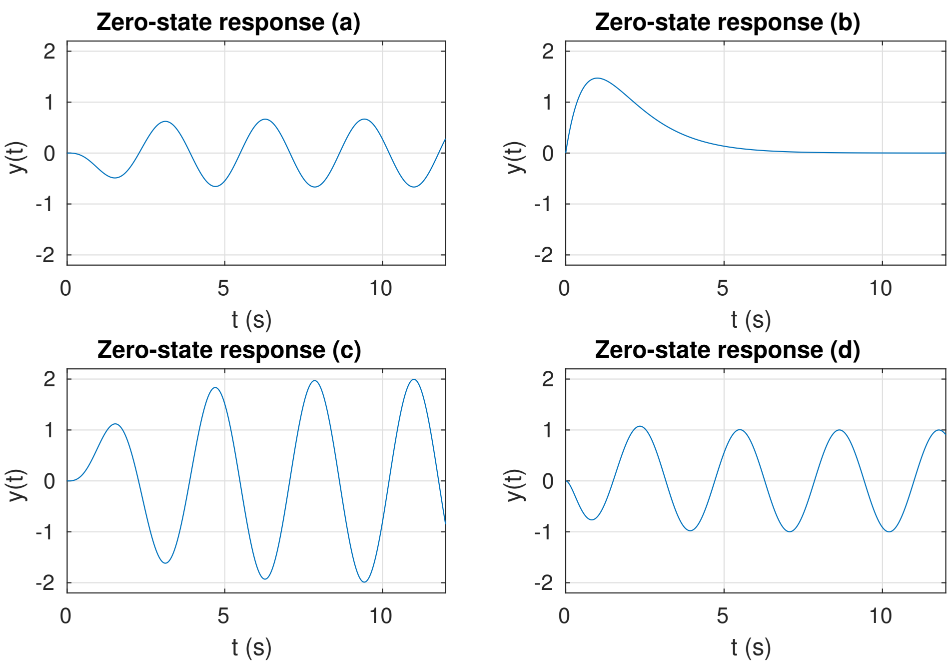

Figure 1: Input signal $u(t)$ and zero-state-response.

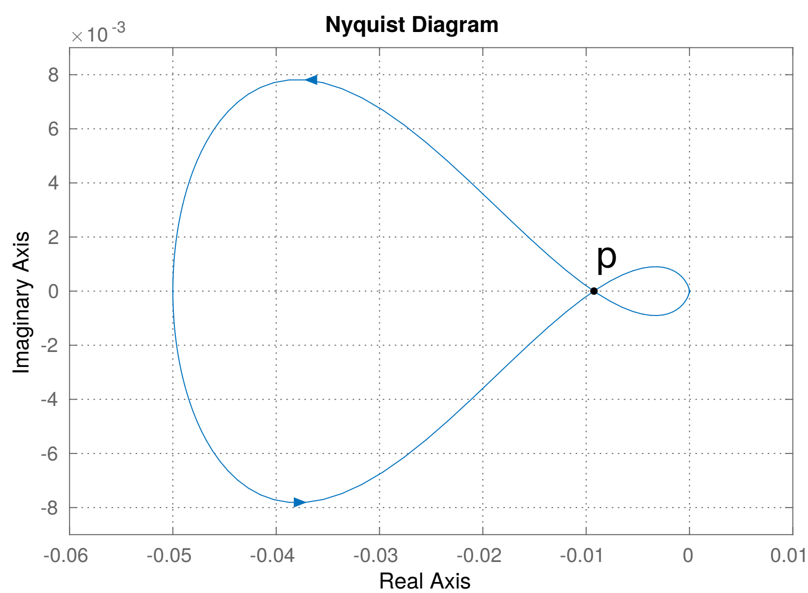

Figure 2: Nyquist diagram of the open-loop transfer function of Σ_1.