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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	Exercise
4	5	8	8	**25 Points**

Consider the system

$$\dot{x}_1 = f_1 = -x_2 e^{x_1 x_2}, \\ \dot{x}_2 = f_2 = x_1 + k e^{x_2}. \tag{1}$$

1. The dimension of the system is 2, since it comprises of two states. It is nonlinear because of the exponential terms, autonomous since there is no input, and time invariant since it does not depend explicitly on time.

2. To compute the equilibrium points consider $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$. The former leads to $x_2 = 0$, whereas the latter to $x_1 = -k$. Hence, the only equilibrium point of the system is $(\hat{x}_1, \hat{x}_2) = (-k, 0)$.

3. The linearized matrix of the system is

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -x_2^2 e^{x_1 x_2} & -e^{x_1 x_2}(1 + x_1 x_2) \\ 1 & k e^{x_2} \end{bmatrix}.$$

Evaluating A at the equilibrium point $(\hat{x}_1, \hat{x}_2)$, we get $A_{(\hat{x}_1,\hat{x}_2)} = \begin{bmatrix} 0 & -1 \\ 1 & k \end{bmatrix}$. To comment on the stability of $(\hat{x}_1, \hat{x}_2)$, the eigenvalues of $A_{(\hat{x}_1,\hat{x}_2)}$ need to be computed. Hence, $\det|\lambda I - A_{(\hat{x}_1,\hat{x}_2)}| = \det|\begin{bmatrix} \lambda & 1 \\ -1 & \lambda - k \end{bmatrix}| = \lambda^2 - k\lambda + 1 = 0$ leads to

$$\lambda_{1,2} = \frac{k \pm \sqrt{k^2 - 4}}{2}.$$

If $k > 0$, both eigenvalues are positive or have positive real parts, and hence $(\hat{x}_1, \hat{x}_2)$ is unstable. If $k < 0$, both eigenvalues are negative or have negative real parts, and hence $(\hat{x}_1, \hat{x}_2)$ is stable. Finally, if $k = 0$, the linearized system has imaginary eigenvalues ($\lambda_{1,2} = \pm j$), and so the linearization is inconclusive.

4. If $k = 0$, the origin $(0, 0)$ is the only equilibrium of the system. As shown in (3), linearization is inconclusive in the case where $k = 0$. To comment on the stability of $(0, 0)$ consider the candidate Lyapunov function $V(x_1, x_2) = x_1^2 + x_2^2$. We have that $V(0, 0) = 0$, and $V(x_1, x_2) > 0$ for all $(x_1, x_2) \neq (0, 0)$. Moreover,

$$\begin{aligned} \frac{dV(x_1, x_2)}{dt} &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= -2x_1x_2e^{x_1x_2} + 2x_1x_2 = 2x_1x_2(1 - e^{x_1x_2}) \leq 0. \end{aligned}$$

The last inequality holds since, based on the hint, $\alpha(1 - e^{\alpha}) \leq 0$ for all $\alpha \in \mathbb{R}$. Hence, $(0, 0)$ is stable but not asymptotically stable, since the last inequality is not strict.

Exercise 2

1	2	3	4	Exercise
5	6	8	6	25 Points

1. The system is nonlinear due to the square term on $\theta(k)$ and has dimension 2. The general state space form of the system can be written as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} x_2(k) \\ -1.5x_2(k) - 3x_1(k) + x_1(k)^2 + u(k) + v(k) \end{bmatrix}$$
$$y(k) = x_2(k)$$

2. The nonlinear system has two equilibrium points. The system is in equilibrium when $x(k+1) = x(k)$, hence we solve the system of equations

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} x_2(k) \\ -1.5x_2(k) - 3x_1(k) + x_1(k)^2 \end{bmatrix}.$$

From the first equation we obtain the relation $x_1(k) = x_2(k)$. Plugging this into the second equation we obtain the quadratic equation

$$x_2(k) = -4.5x_2(k) + x_2(k)^2.$$

Solving the quadratic equation we obtain solutions $x_2(k) = 0$ and $x_2(k) = 5.5$. Hence, the two equilibrium points are $x(k) = [0, 0]^T$ and $x(k) = [5.5, 5.5]^T$.

3. For the system to have a unique equilibrium at the origin, the constant $a = 0$. For the system to be linear, the nonlinearity must be canceled completely, thus the constant $c = -1$. Applying the resulting feedback law $v(k) = bx_1(k) - x_1(k)^2$ to the nominal system we obtain a linear system of the form $x(k+1) = Ax(k)$ with state matrix

$$A = \begin{bmatrix} 0 & 1 \\ -3+b & -1.5 \end{bmatrix}.$$

The characteristic equation of A is

$$s^2 + 1.5x + 3 - b = 0.$$

Considering that a system with eigenvalues at -0.5 and -1 must satisfy the characteristic equation

$$s^2 + 1.5s + 0.5 = 0,$$

we conclude that the constant $b = 2.5$. Hence, the feedback control law is

$$v(k) = 2.5x_1(k) + x_1(k)^2.$$

The system is stable but not asymptotically stable since the absolute value of one eigenvalue ($s = -1$) is not strictly less than 1.

4. With the input $v(k)$ defined as above, the resulting discrete time linear system takes the form

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} 0 & 1 \\ -0.5 & -1.5 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\
y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(k)
\end{aligned}
$$

The observability matrix

$$
Q = \begin{bmatrix} 0 & 1 \\ -0.5 & -1.5 \end{bmatrix}
$$

has full rank, thus the system is observable. The controllability matrix

$$
P = \begin{bmatrix} 0 & 1 \\ 1 & -1.5 \end{bmatrix}
$$

has full rank, thus the system is controllable.

Exercise 3

1	2	3	Exercise
9	7	9	25 Points

1. To directly compute $\Phi(t)$, decompose A as

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{A_1} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A_2} \quad .$$

Since A_1 is diagonal it commutes with any other square matrix, i.e., $A_1 M = M A_1$ for any $M \in \mathbb{R}^{3\times 3}$. Therefore, $e^{At} = e^{A_1 t} e^{A_2 t}$, cf. Examples Set 3, Exercise 3.

For the diagonal matrix A_1 the matrix exponential is easy to compute:

$$e^{A_1 t} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

Since A_2 is nilpotent, the power series expansion of its matrix exponential is in fact finite with

$$A_2^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $A_2^3 = 0$. Accordingly,

$$e^{A_2 t} = I + A_2 t + \frac{1}{2} A_2^2 t^2 = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\Phi(t) = e^{At} = e^{A_1 t} e^{A_2 t} = e^{-t} \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \quad .$$

Alternatively, by differentiating $\Phi(t)$ to

$$\frac{d}{dt}\Phi(t) = \dot{\Phi}(t) = e^{-t} \begin{bmatrix} -1 & 1-t & t-\frac{t^2}{2} \\ 0 & -1 & 1-t \\ 0 & 0 & -1 \end{bmatrix} \quad ,$$

it can be shown that

$$\dot{\Phi}(t) = A\Phi(t) \quad .$$

Accordingly, $\Phi(t)$ is indeed the state transition matrix corresponding to A.

Since A is in Jordan-Normal-Form, the eigenvalues can be read off the diagonal and $\lambda = -1$ with algebraic multiplicity 3 and geometric multiplicity 1. Therefore, A is not diagonalizable.

Since $\text{Re}\{\lambda\} < 0$, from Theorem 3.2 the system is asymptotically stable and therefore also stable.

The system is asymptotically stable, see previous solution.

2. Since A is the zero-matrix, the power series expansion of the matrix exponential only contains the constant element. Accordingly,

$$\Phi(t) = e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad .$$

Since A is diagonal (and therefore diagonalizable), the eigenvalues can be read off the diagonal and $\lambda = 0$ with algebraic multiplicity 3 and geometric multiplicity 3.

Since $\text{Re}\{\lambda\} \leq 0$ and A is diagonalizable, the system is stable by Theorem 3.1.

Because the only eigenvalue $\lambda = 0$ (and not strictly smaller than 0), the system is not asymptotically stable (Theorem 3.1).

3. Note that A here is equal to A_2 from the solution of 1.1. Therefore,

$$\Phi(t) = e^{At} = I + At + \frac{1}{2}A^2t^2 = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

From the fact that A is in Jordan-Normal-Form, $\lambda = 0$ with algebraic multiplicity 3 and geometric multiplicity 1.

Since A is not diagonalizable, Theorem 3.1 does not apply. Therefore, establishing stability is a little more involved. From $\Phi(t)$ it can be seen that the solution of $x(t)$ will include terms such as t and t^2 which grow to infinity over time, cf. Notes 3.31f. Accordingly, the system is generally not stable.

From Theorem 3.2 it can be seen that the system is not asymptotically stable. Negative real parts of all eigenvalues are a necessary and sufficient condition for asymptotic stability.

Exercise 4

1	2	3	4	5	Exercise
7	4	4	4	6	25 Points

1. We define the states of the system as the voltages over the capacitors, i.e.,

$$x(t) = \begin{bmatrix} u_{C_1}(t) & u_{C_1}(t) \end{bmatrix}^T .$$

Because of the two energy storages (capacitors) this is clearly a system of second order.

From Kirchhoff's voltage law for the first loop, we get the expressions

$$u(t) = u_{R_1}(t) + u_{C_1} + u_{C_2} \tag{2}$$

and from the current law

$$i_{R_1}(t) = i_{C_1}(t) \Leftrightarrow \frac{1}{R_1}(u(t) - u_{C_1} - u_{C_2}) = C_1 \frac{\mathrm{d}u_{C_1}(t)}{\mathrm{d}t} \tag{3}$$

which gives the differential equation for $x_1(t)$

$$\frac{\mathrm{d}u_{C_1}(t)}{\mathrm{d}t} = \frac{u(t) - u_{C_1} - u_{C_2}}{R_1 C_1} \tag{4}$$

Similar, by substituting the currents of the second loop

$$i_{R_1}(t) = i_{R_2}(t) + i_{C_1}(t) \Leftrightarrow \frac{1}{R_1}(u(t) - u_{C_1} - u_{C_2}) = \frac{u_{R_2}(t)}{R_2} + C_2 \frac{\mathrm{d}u_{C_2}(t)}{\mathrm{d}t} \tag{5}$$

we obtain for $x_2(t)$

$$\frac{\mathrm{d}u_{C_2}(t)}{\mathrm{d}t} = \frac{u(t) - u_{C_1} - u_{C_2}}{R_1 C_2} - \frac{u_{R_2}(t)}{C_2 R_2} \tag{6}$$

Thus, the state space representation with the input $u(t)$ and output $y(t)$

$$\begin{aligned} \dot{x} &= \begin{bmatrix} \frac{-1}{R_1 C_1} & \frac{-1}{R_1 C_1} \\ \frac{-1}{R_1 C_2} & -\frac{R_1 + R_2}{R_1 R_2 C_2} \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{R_1 C_1} \\ \frac{1}{R_1 C_2} \end{bmatrix} u(t) \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + 0u(t) \end{aligned} \tag{7}$$

2. Using the given values for C_i and R_i we compute the contrallability matrix

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix}, \tag{8}$$

and see that the system is controllable, since the matrix has full rank.
The observability matrix shows that the system is observable.

$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \tag{9}$$

Stability can be shown by looking at the eigenvalues of the system.

$$\begin{aligned}\det(\lambda I - A) &= \det\begin{bmatrix}\lambda+1 & 1\\ 1 & \lambda+2\end{bmatrix}\\ &= (\lambda+1)(\lambda+2)-1 = \lambda^2+3\lambda+1\\ &\Rightarrow \lambda_{1/2} = -\frac{3}{2}\pm\frac{\sqrt{5}}{2}\end{aligned} \tag{10}$$

Both eigenvalues are negative, hence the system is stable and asymptotically stable as well.

3. With the matrices from part 1, the controllability matrix is given by

$$P = \begin{bmatrix}\frac{1}{R_1C_1} & -\frac{1}{(R_1C_1)^2}-\frac{1}{R_1^2C_1C_2}\\ \frac{1}{R_1C_2} & -\frac{1}{R_1^2C_1C_2}-\frac{R_1+R_2}{R_1^2R_2C_2^2}\end{bmatrix}. \tag{11}$$

Under the assumption $C_1 = C_2 = C$ we loose controllability if the last terms of the second column are equal, i.e.,

$$\begin{aligned}\frac{1}{R_1^2C^2} &= \frac{R_1+R_2}{R_1^2R_2C^2}\\ \Leftrightarrow 1 &= \frac{R_1+R_2}{R_2},\end{aligned} \tag{12}$$

which holds approximately for large $R_2 \gg R_1$.

4. The transfer function can be computed directly form the system matrices.

$$\begin{aligned}G(s) &= C(s\mathbf{I}-A)^{-1}B\\ &= \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}s+1 & 1\\ 1 & s+2\end{bmatrix}^{-1}\begin{bmatrix}1\\1\end{bmatrix}\\ &= \frac{1}{s^2+3s+1}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}s+2 & -1\\ -1 & s+1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\\ &= \frac{s}{s^2+3s+1}.\end{aligned}$$

5. Given the controller $u(t) = \begin{bmatrix}k_1 & k_2\end{bmatrix} x(t)$ we can rewrite the system dynamics as an autonomous system

$$\dot{x} = A + BKx(t) = \begin{bmatrix}k_1-1 & k_2-1\\ k_1-1 & k_2-2\end{bmatrix} x(t). \tag{13}$$

Computing the characteristical polynomial of the closed loop system, we solve

$$\det(I\lambda - A - BK) = \det\begin{bmatrix}\lambda-k_1+1 & -k_2+1\\ -k_1+1 & \lambda-k_2+2\end{bmatrix} \tag{14}$$

$$= \lambda^2 - \lambda(k_1+k_2-3)+1-k_1 \tag{15}$$

To place the poles, we need to match the coefficients of the characteristic polynomial of the system with the desired one. For both poles at -1 we get the desired polynomial $\lambda^2 + 2\lambda + 1$ and thus the conditional equations

$$\begin{aligned} 2 &= 3 - k_1 - k_2 \\ 1 &= 1 - k_1. \end{aligned}$$

From the second we get $k_1 = 0$ and therefore obtain $k_2 = 1$.