Automatic Control Laboratory
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Signal and System Theory II
4. Semester, BSc

Solutions

1 Exercise 1

1	2	3	4	Aufgabe
5	5	10	5	25 Punkte

1. Controllability matrix:

$$\begin{aligned} \mathcal{C} &= \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ 1 & 2 \end{bmatrix} \\ \det \mathcal{C} &= -\alpha \end{aligned}$$

   If system is controllable, we need to have

$$\det(\mathcal{C}) \neq 0 \rightarrow \alpha \neq 0$$

   The system is controllable for all $\alpha \in \mathbb{R}\backslash 0$.

2. Observability matrix:

$$\begin{aligned} \mathcal{O} &= \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & \alpha \end{bmatrix} \\ \det \mathcal{O} &= \alpha - 1 \end{aligned}$$

   If system is observable, we need to have

$$\det(\mathcal{O}) \neq 0 \rightarrow \alpha \neq 1$$

   The system is observable for all $\alpha \in \mathbb{R}\backslash 1$.

3. Compute characteristic polynomial for poles at -1 and -7:

$$(\lambda + 1)(\lambda + 7) = \lambda^2 + 8\lambda + 7$$

   With the feedback controller the resulting closed loop system is

$$\begin{aligned} A^* &= (A + BK) = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ k_1 & 2 + k_2 \end{bmatrix} \\ \det(A^*) &= (1 - \lambda)(2 + k_2 - \lambda) - 5k_1 \\ &= \lambda^2 + (-3 - k_2)\lambda + 2 + k_2 - 5k_1 \end{aligned}$$

   Comparison of coefficients:

$$\begin{aligned} -3 - k_2 &= 8 \\ \wedge \quad +2 + k_2 - 5k_1 &= 7 \\ \Rightarrow k_1 = -\frac{16}{5}, \qquad k_2 &= -11 \end{aligned}$$

4. The closed loop system is stable since we designed it to have the poles in the left half-plane.

2 Exercise 2

1	2	3	4	Exercise
5	5	9	6	25 Points

1. In state space form, the discrete time system can be expressed:

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ \frac{\nu+1}{m} & -\frac{d}{m} \end{bmatrix} x_k + \begin{bmatrix} 0 \\ \frac{b}{m} \end{bmatrix} u_k.$$

2. The dimension of the system is 2. The system is not autonomous since it has an input. The system is linear.

3. For $\nu = -1$, the state matrix is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix}.$$

The eigenvalues are $\lambda = 0, \ -0.5$ and therefore the system is stable.

For $\nu = 0$, the state matrix is

$$A = \begin{bmatrix} 0 & 1 \\ 0.5 & -0.5 \end{bmatrix}.$$

The eigenvalues are $\lambda = 0.5, \ -1$ and therefore the system is marginally stable.

For $\nu = 2$, the state matrix is

$$A = \begin{bmatrix} 0 & 1 \\ 1.5 & -0.5 \end{bmatrix}.$$

The eigenvalues are $\lambda = 1, \ -1.5$ and therefore the system is unstable.

4. Setting $u_k = \begin{bmatrix} -1 & \frac{1}{3} \end{bmatrix} x_k$, the system becomes:

$$x_{k+1} = \left(\begin{bmatrix} 0 & 1 \\ 1.5 & -0.5 \end{bmatrix} + \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} \begin{bmatrix} -1 & \frac{1}{3} \end{bmatrix}\right) x_k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_k.$$

The system matrix is nilpotent and thus $x_k = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ for all $k \geq 2$.

3 Exercise 3

1	2	3	Exercise
10	7	8	25 Points

1. Based on Kirchhoff's laws, we have that

$$V_i = V_{L_1} + V_{L_2} \tag{1}$$
$$= L_1 \frac{di_{L_1}}{dt} + L_2 \frac{di_{L_2}}{dt}. \tag{2}$$

We also have that

$$i_{R_1} + i_{L_1} = i_{R_2} + i_{L_2}.$$

Hence

$$\frac{L_1}{R_1}\frac{di_{L_1}}{dt} + i_{L_1} = \frac{L_2}{R_2}\frac{di_{L_2}}{dt} + i_{L_2}.$$

From the above two equations we conclude that the state space representation of the system is

$$\frac{d}{dt}\begin{bmatrix} i_{L_1} \\ i_{L_2} \end{bmatrix} = \begin{bmatrix} -\frac{R_1R_2}{L_1(R_1+R_2)} & \frac{R_1R_2}{L_1(R_1+R_2)} \\ \frac{R_1R_2}{L_2(R_1+R_2)} & -\frac{R_1R_2}{L_2(R_1+R_2)} \end{bmatrix}\begin{bmatrix} i_{L_1} \\ i_{L_2} \end{bmatrix} + \begin{bmatrix} \frac{R_1}{L_1(R_1+R_2)} \\ \frac{R_2}{L_2(R_1+R_2)} \end{bmatrix} V_i.$$

From the output of the system we have that $V_0 = V_{L_2} = L_2\frac{di_{L_2}}{dt}$, which is

$$V_0 = \begin{bmatrix} \frac{R_1R_2}{R_1+R_2} & -\frac{R_1R_2}{R_1+R_2} \end{bmatrix}\begin{bmatrix} i_{L_1} \\ i_{L_2} \end{bmatrix} + \frac{R_2}{R_1+R_2}V_i.$$

2. Consider the controllability matrix $P = [B\ AB]$. For $R_1 = R_2 = 1\Omega$, and $L_1 = L_2 = 0.5H$ we have that

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then the controllability matrix is

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

We obtain that $rank(P) = 1$, since the rows of P are linearly dependent, and hence the system is uncontrollable.

Similarly, by checking the observability matrix $Q = \begin{bmatrix} C \\ CA \end{bmatrix}$, we have that

$$Q = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 1 \end{bmatrix}.$$

We obtain that $rank(Q) = 1$, and hence the system is unobservable.

3. For the case where $V_i = 0$ the system is reduced to

$$\frac{d}{dt}\begin{bmatrix} i_{L_1} \\ i_{L_2} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} i_{L_1} \\ i_{L_2} \end{bmatrix}.$$

The response of the system is given by $x(t) = \Phi(t)x(0)$, where $\Phi(t) = e^{At}$ is the state transition matrix. From $det(\lambda I - A) = 0$, the eigenvalues of A are calculated as $\lambda_1 = 0, \lambda_2 = -2$. Denote as w_1, w_2 the eigenvectors that correspond to the eigenvalues λ_1 and λ_2 respectively. Since $\tilde{A}w_i = \lambda_i w_i$ for $i = 1, 2$, the eigenvectors are calculated as

$$w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence the eigenvectors matrix is $W = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. The response of the system is given by $x(t) = \Phi(t)x(0)$, where $\Phi(t) = e^{At} = We^{\Lambda t}W^{-1}$, and $\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$. Then,

$$\begin{aligned}\begin{bmatrix} i_{L_1} \\ i_{L_2} \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-2t} \end{bmatrix}\frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}.\end{aligned}$$

4 Exercise 4

1	2	3	Exercise
8	9	8	25 Points

1. Since both systems are controllable, the controllability matrices $\mathcal{C}_1 = [B \;\; A_1 B]$ and $\mathcal{C}_2 = [B \;\; A_2 B]$ are both full rank. This, though, cannot indicate anything for $\mathcal{C}_3 = [B \;\; (A_1 + A_2)B]$. One can see that easily by considering the two cases where $A_2 = A_1$ and $A_2 = -A_1$. In the former, we have $\mathcal{C}_3 = [B \;\; 2A_1 B] \Rightarrow \det \mathcal{C}_3 = 2 \det \mathcal{C}_1 \neq 0$, i.e. system (3) is controllable and in the latter, $\mathcal{C}_3 = [B \;\; \mathbb{O}B] \Rightarrow \det \mathcal{C}_3 = 0$, i.e. system (3) is uncontrollable.

2. Let $A_1 = \begin{bmatrix} a_1^1 & a_2^1 \\ a_3^1 & a_4^1 \end{bmatrix}$, $A_2 = \begin{bmatrix} a_1^2 & a_2^2 \\ a_3^2 & a_4^2 \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. We know that $\det \mathcal{C}_2 = 0$, i.e.:

$$\begin{vmatrix} b_1 & a_1^2 b_1 + a_2^2 b_2 \\ b_2 & a_3^2 b_1 + a_4^2 b_2 \end{vmatrix} = b_1(a_3^2 b_1 + a_4^2 b_2) - b2(a_1^2 b_1 + a_2^2 b_2) = 0.$$

   Now, calculating $\det \mathcal{C}_3$, we get:

$$\begin{aligned} \det \mathcal{C}_3 &= \begin{vmatrix} b_1 & (a_1^1 + a_1^2) b_1 + (a_2^1 + a_2^2) b_2 \\ b_2 & (a_3^1 + a_3^2) b_1 + (a_4^1 + a_4^2) b_2 \end{vmatrix} \\ &= b_1[(a_3^1 + a_3^2) b_1 + (a_4^1 + a_4^2) b_2] - b2[(a_1^1 + a_1^2) b_1 + (a_2^1 + a_2^2) b_2] \\ &= b_1(a_3^1 b_1 + a_4^1 b_2) - b2(a_1^1 b_1 + a_2^1 b_2) + b_1(a_3^2 b_1 + a_4^2 b_2) - b2(a_1^2 b_1 + a_2^2 b_2) \\ &= \det \mathcal{C}_1 + \det \mathcal{C}_2 = \det \mathcal{C}_1 \neq 0. \end{aligned}$$

   Thus, system (3) in this case is controllable.

3. Nothing can be said for system (4). Take for instance any invertible matrix A_1 such that system (1) is controllable. Then, taking $A_2 = A_1^{-1}$, one always gets an uncontrollable system (4). On the contrary, one can easily find a matrix A_2 that makes system (2) controllable, as well as system (4).