Automatic Control Laboratory
ETH Zurich
Prof. J. Lygeros

D-ITET
Winter 2009
26.01.2009

Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	Aufgabe
8	6	11	25 Punkte

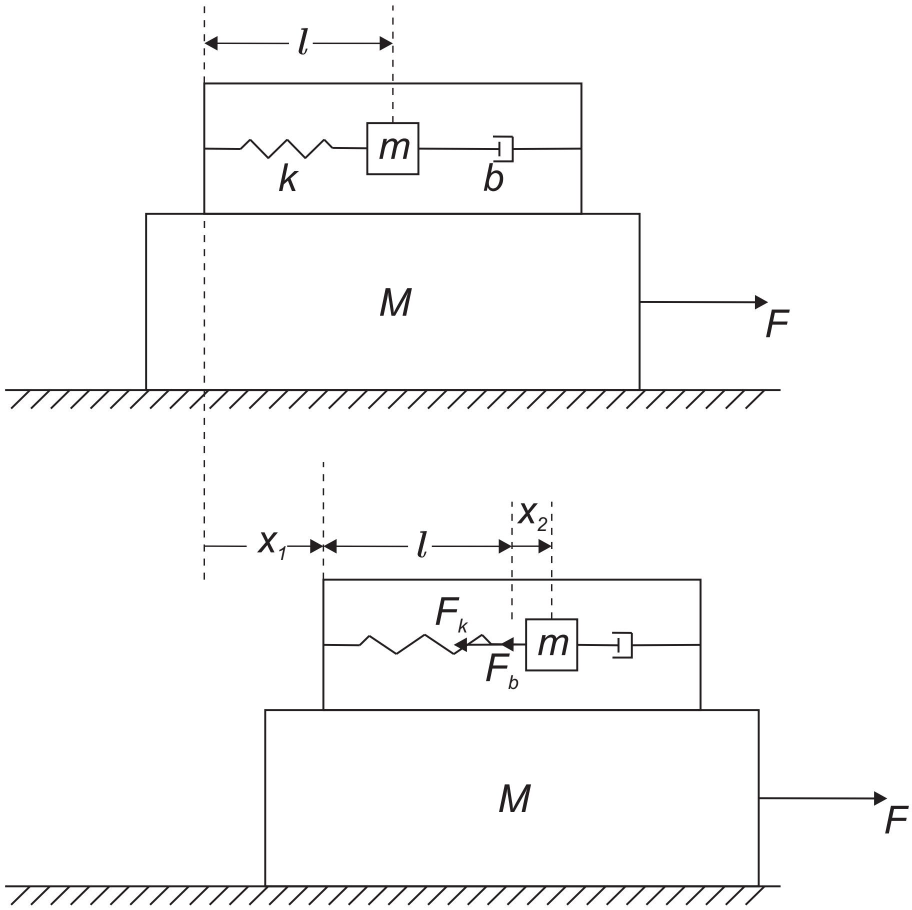

Figure 1: A mechanical accelerometer mounted on a mass M.

1. Using Newton's law of motion we have:

   Mass M:

$$\begin{aligned} \sum F &= Ma \\ F &= M\frac{d^2x_1}{dt^2} \\ \ddot{x}_1 &= \frac{u}{M} \end{aligned} \tag{1}$$

   Mass m:

$$\begin{aligned}
\sum F &= ma \\
-kx_2 - b\frac{dx_2}{dt} &= m\frac{d^2}{dt^2}(x_1 + x_2) \\
-kx_2 - b\frac{dx_2}{dt} &= m\ddot{x}_1 + m\ddot{x}_2 \\
\ddot{x}_2 &= -\frac{k}{m}x_2 - \frac{b}{m}\dot{x}_2 - \frac{u}{M} \qquad (2)
\end{aligned}$$

Using $z_1 = x_1$, $z_2 = \dot{x}_1$, $z_3 = x_2$ and $z_4 = \dot{x}_2$ as states, the state space model is:

$$\begin{bmatrix} \frac{dz_1}{dt} \\ \frac{dz_2}{dt} \\ \frac{dz_3}{dt} \\ \frac{dz_4}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{M} \end{bmatrix} u \qquad (3)$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \qquad (4)$$

2. The output of the system is $y = x_2$. Therefore from equation (??) and using the Laplace transform we have:

$$\begin{aligned}
s^2Y(s) &= -\frac{k}{m}Y(s) - \frac{b}{m}sY(s) - \frac{1}{M}U(s) \\
(s^2 + \frac{b}{m}s + \frac{k}{m})Y(s) &= -\frac{1}{M}U(s) \\
G(s) = \frac{Y(s)}{U(s)} &= -\frac{1}{M}\frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \qquad (5)
\end{aligned}$$

3. Replacing the given values we get the transfer function:

$$G(s) = -\frac{1}{5}\frac{1}{s^2 + 4s + 4} = -\frac{1}{5(s+2)^2}$$

The Laplace transform of the output $y(t)$ is:

$$Y(s) = -\frac{1}{5(s+2)^2}\frac{10}{s} = -\frac{2}{s(s+2)^2}$$

The time expression of the output $y(t)$ can be found using inverse Laplace transform. We start be expanding the transfer function in partial fractions.

$$
\begin{aligned}
-\frac{2}{s(s+2)^2} &= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2} \\
-\frac{2}{s(s+2)^2} &= \frac{A(s^2+4s+4) + B(s^2+2s) + Cs}{s(s+2)^2} \\
-\frac{2}{s(s+2)^2} &= \frac{(A+B)s^2 + (4A+2B+C)s + 4A}{s(s+2)^2}
\end{aligned}
$$

Equating coefficients, we obtain $A = -0.5$, $B = 0.5$ and $C = 1$. Therefore the Laplace transform of the output is:

$$Y(s) = -\frac{0.5}{s} + \frac{0.5}{s+2} + \frac{1}{(s+2)^2}$$

and the inverse Laplace transform is:

$$y(t) = -0.5 + 0.5e^{-2t} + te^{-2t} \tag{6}$$

Exercise 2

1	2	3	4	Exercise
5	5	5	10	25 Points

1. To check the controllability, we compute the controllability matrix

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & a+1 \\ 1 & -3 \end{bmatrix}. \tag{7}$$

The determinant of P is $det(P) = -a-1$, therefore P has full rank whenever $a \neq -1$.

2. To check the observability, we compute the observability matrix

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & a-2 \end{bmatrix}. \tag{8}$$

The determinant of Q is $det(Q) = a-3$, therefore Q has full rank whenever $a \neq 3$.

3. We begin by evaluating the characteristic polynomial by taking the determinant of $\lambda I - A$ where I is the identity matrix.

$$\begin{aligned} det(\lambda I - A) &= det\begin{pmatrix} \lambda & -a-1 \\ -1 & \lambda+3 \end{pmatrix} &(9)\\ &= \lambda^2 + 3\lambda - a - 1. &(10)\end{aligned}$$

We know that a second order system is stable if and only if all coefficients of the quadratic characteristic equation have the same sign. Therefore, we have that the system is asymptotically stable for $a < -1$, and unstable for $a > -1$. For $a = 1$ the characteristic polynomial becomes $\lambda(\lambda+3)$, with the corresponding eigenvalues being $\lambda_1 = 0$ and $\lambda_2 = -3$. Hence, the system in this case is stable, but not asymptotically stable.

4. We wish to design a feedback controller which places the closed loop poles at $\lambda = -1$. Given the state feedback controller, $u = Kx$, we may rewrite the systems as an autonomous system, i.e.

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & a+1 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} x &(11)\\ &= \begin{bmatrix} 0 & a+1 \\ 1+k_1 & -3+k_2 \end{bmatrix} x. &(12)\end{aligned}$$

Computing the characteristic polynomial of the closed loop system, we take

$$\begin{aligned} det(\lambda I - A - BK) &= det\begin{pmatrix} \lambda & -a-1 \\ -1-k_1 & \lambda+3-k_2 \end{pmatrix} &(13)\\ &= \lambda^2 + (3-k_2)\lambda - (a+1)(k_1+1). &(14)\end{aligned}$$

To place the poles at $\lambda = -1$, we simply need to match the coefficients of the above characteristic polynomial to the polynomial with roots at -1, i.e. $\lambda^2 + 2\lambda + 1$. Therefore, we can first solve for k_2 by solving $2 = 3 - k_2$, which gives us $k_2 = 1$. Next, we try to solve for k_1 by solving $1 = -(a+1)(k_1+1)$. This is impossible if $a = -1$, otherwise the solution is $k_1 = -\frac{2+a}{1+a}$. Therefore, if $a \neq -1$, our controller is

$$K = \begin{bmatrix} -\frac{2+a}{1+a} & 1 \end{bmatrix}. \tag{15}$$

Notice that the controller K is undefined at $a = -1$, and therefore the pole placement is not feasible. Of course, this is consistent with part 1) where we noted that the systems is uncontrollable at $a = -1$.

Exercise 3

1	2	3	4	Aufgabe
5	5	10	5	25 Punkte

1. Setting $x_1 = \theta$ and $x_2 = \dot{\theta}$ we can derive the following equations:

$$\dot{x}_1 = x_2 \tag{16}$$
$$\dot{x}_2 = \sin(-x_1 + \alpha x_2) \tag{17}$$

2. In order to determine the equilibrium points of the above system we set the derivatives in equations (**??**),(**??**) equal to zero. So we have:

$$x_2^\star = 0 \tag{18}$$
$$\sin(-x_1^\star + \alpha x_2^\star) = 0 \tag{19}$$

Substituting (**??**) into (**??**) we obtain that $\sin(-x_1^\star) = 0$, so $x_1^\star = -k\pi$, for $k \in \mathbb{Z}$. Equilibria for even values of k lead to the same system behavior, so for stability we can check $x_a = (0,0)$ as a representative of the points obtained for $k = 0, 2, 4, ...$ The same hold for odd values of k, so we can check only $x_b = (-\pi, 0)$.

3. The Jacobian of the system is:

$$A(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ -\cos(-x_1 + \alpha x_2) & \alpha\cos(-x_1 + \alpha x_2) \end{bmatrix} \tag{20}$$

For all equilibria of the form $(0, 2k\pi)$, for $k \in \mathbb{Z}$, the Jacobian becomes $A = \begin{bmatrix} 0 & 1 \\ -1 & \alpha \end{bmatrix}$. The characteristic polynomial is therefore $\lambda^2 - \alpha\lambda + 1$.

- If $\alpha > 0$ the coefficients of the characteristic polynomial have different signs. This implies that a root has positive real part. Therefore the system is unstable.
- If $\alpha < 0$ the coefficients of the characteristic polynomial have same sign. This implies that the roots have negative real part. Therefore the system is stable.
- If $\alpha = 0$ the roots of the characteristic polynomial are imaginary. Therefore the linearization is inconclusive.

For all equilibria of the form $(0, (2k+1)\pi)$, for $k \in \mathbb{Z}$ the Jacobian becomes $A = \begin{bmatrix} 0 & 1 \\ 1 & -\alpha \end{bmatrix}$. The characteristic polynomial of this matrix is $\lambda^2 + \alpha\lambda - 1$. Hence the system is unstable for all values of α.

4. For $\alpha = 0$ the state equations become:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin(-x_1) = -\sin(x_1) \end{aligned}$$

This is a special case of the frictionless pendulum threated in class. For $V(x) = (1 - cos(x_1)) + \frac{1}{2}x_2^2$ we have that $V(0) = 0$ and $V(x) > 0$ for $x \in \mathbb{R}^2$ with $|x_1| < 2\pi$. Moreover,

$$\begin{aligned} \dot{V}(x) &= \sin(x_1)x_2 + x_2(-\sin(x_1)) \\ &= 0 \end{aligned}$$

Applying Lyapunov's Stability Theorem to the open set

$$S = \{x \in \mathbb{R}^2 | |x_1| < 2\pi\}$$

we conclude that the equilibrium $x = 0$ is stable.

Exercise 4

1	2	3	Aufgabe
5	10	10	25 Punkte

1. If A is diagonalizable then it can be written in the form $A = S\Lambda S^{-1}$. We can use induction for the proof. So, for k=1, $A^1 = S\Lambda^1 S^{-1}$ holds by definition. Assume that

$$A^{k-1} = S\Lambda^{k-1}S^{-1}$$

and show that $A^k = S\Lambda^k S^{-1}$. This is clearly true since

$$\begin{aligned} A^k &= A^{k-1}A \\ &= S\Lambda^{k-1}S^{-1}S\Lambda S^{-1} \\ &= S\Lambda^{k-1}\Lambda S^{-1} \\ &= S\Lambda^k S^{-1} \end{aligned}$$

2. We start by calculating the product of the first two matrices:

$$P = S\Lambda^k = \begin{bmatrix} & \vdots & & \vdots & \dots & \vdots & \\ s_1 & \vdots & s_2 & \vdots & \dots & \vdots & s_n \\ & \vdots & & \vdots & \dots & \vdots & \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix}$$

which leads to

$$P = s_1[\lambda_1^k\ 0\ 0\ \dots\ 0] + s_2[0\ \lambda_2^k\ 0\ \dots\ 0] + \dots + s_n[0\ 0\ 0\ \dots\ \lambda_n^k]$$

We can observe that this is a sum of n matrices, each of dimension $n \times n$. For the first matrix, we have:

$$s_1[\lambda_1^k\ 0\ 0\ \dots\ 0] = \begin{bmatrix} s_{11}\lambda_1^k & 0 & \dots & 0 \\ s_{12}\lambda_1^k & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ s_{1n}\lambda_1^k & 0 & \dots & 0 \end{bmatrix}$$

$$= \lambda_1^k \begin{bmatrix} s_{11} & 0 & \dots & 0 \\ s_{12} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ s_{1n} & 0 & \dots & 0 \end{bmatrix} \qquad = \lambda_1^k \begin{bmatrix} & \vdots & & \vdots & \dots & \vdots & \\ s_1 & \vdots & 0 & \vdots & \dots & \vdots & 0 \\ & \vdots & & \vdots & \dots & \vdots & \end{bmatrix}$$

where $s_{11}, s_{12}, ...s_{1n}$ are the individual elements of the eigenvector s_1. Similar relations hold for the rest of the matrices, so finally we get:

$$P = \lambda_1^k \left[\begin{array}{c:c:c:c} & \vdots & \vdots \;...\; \vdots & \\ s_1 & 0 & ... & 0 \\ & \vdots & \vdots \;...\; \vdots & \end{array} \right] + ... + \lambda_n^k \left[\begin{array}{c:c:c:c} & \vdots & \vdots \;...\; \vdots & \\ 0 & 0 & ... & s_n \\ & \vdots & \vdots \;...\; \vdots & \end{array} \right]$$

$$= \left[\begin{array}{c:c:c:c} \vdots & & \vdots \;...\; \vdots & \\ \lambda_1^k s_1 & \lambda_2^k s_2 & ... & \lambda_n^k s_n \\ \vdots & & \vdots \;...\; \vdots & \end{array} \right]$$

Now, given that

$$A = PS^{-1} = \left[\begin{array}{ccc} \lambda_1^k s_1 & \dots & \lambda_n^k s_n \end{array} \right] \left[\begin{array}{c} e_1^T \\ \vdots \\ e_n^T \end{array} \right]$$

by a similar procedure

$$A = \sum_{i=1}^{n} \lambda_i^k s_i e_i^T$$

3. We can examine the stability of the system using the above relation, since the zero input response of the system is $x_k = A^k x_0$. So,

$$x_k = \left(\sum_{i=1}^{n} \lambda_i^k s_i e_i^T \right) x_0$$

$$\begin{aligned} \|x_k\| &= \| \sum_{i=1}^{n} \lambda_i^k s_i e_i^T x_0 \| \\ &\leq \sum_{i=1}^{n} |\lambda_i^k| \|s_i\| \|e_i\| \|x_0\| \\ &= \sum_{i=1}^{n} |\lambda_i|^k \|s_i\| \|e_i\| \|x_0\| \end{aligned}$$

Recall that λ_i are generally complex. If $|\lambda_i| < 1$ for all $i = 1, ..., n$ then $|\lambda_i|^k \to 0$ as $k \to \infty$, for all $i = 1, ..., n$ since everything else in this expression (s_i, e_i, x_0) are constant $x_k \to 0$ as $k \to \infty$ and the system is asymptotically stable.

If $|\lambda_i| = 1$ for all $i = 1, ..., n$ then $|\lambda^k| = |\lambda_i|^k = 1$ for all $k = 0, ..., 1$ and all $i = 1, ..., n$. Therefore

$$\|x_k\| = \|\sum_{i=1}^{n} \lambda_i^k s_i e_i^T x_0\| \leq \sum_{i=1}^{n} |\lambda_i^k| \|s_i\| \|e_i\| \|x_0\|$$
$$\leq \sum_{i=1}^{n} \|s_i\| \|e_i\| \|x_0\|$$

Hence the state trajectory remains bounded (though it does not necessarily converge to 0) and the system is stable (though not necessarily asymptotically stable).

Finally, if there exists $i = 1, ..., n$ such that $|\lambda_i| > 1$, set $x_0 = s_i$. Since $S^{-1} = \begin{bmatrix} e_1^T \\ \vdots \\ e_n \end{bmatrix}$ and $S^{-1}S = I$, we have $e_i^T x_0 = e_i^T s_i = 1$ and $e_i^T x_0 = e_i^T s_i = 0$ if $j \neq i$. Therefore

$$x_k = \sum_{i=1}^{k} \lambda_i^k s_i e_i^T x_0 = \lambda_i^k s_i$$

and $\|x_k\| = |\lambda_i|^k \|s_i\| \to \infty$ as $k \to \infty$. Hence the system is unstable.